Free Vibration Analysis Of A Pretwisted Functionally Graded Material Cantilever Timoshenko Beam

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Abstract-- In the present work the free vibration of a functionally graded ordinary (FGO) pre-twisted cantilever Timoshenko beam has been investigated. Finite element shape functions are established from differential equations of static equilibrium. Expressions for element stiffness and mass matrices are obtained from energy considerations. The material properties along the thickness of the beam are assumed to vary according to the power law. Increase in the value of power law index decreases the first two mode frequencies of the beam. Increase in pretwist angle increases first mode frequency and decreases the second mode frequency. The effect of pretwist angle and power law index on the first mode frequency is marginal.

Keywords-- FGO; Power law; Pretwist angle.

I. INTRODUCTION

SOME machine components are twisted for functional point of view. The blades of helicopter rotor, turbine and aircraft propeller are given pre-twist and usually are subjected to vibration. The study of free vibration of pre-twisted components therefore remains an important area of research. Carnegie [1, 2] studied the dynamic behaviour of a pre-twisted cantilever blade taking into account torsion, bending rotary inertia and shear deformation. Dawson, Ghosh and Carnegie [3] investigated the effect of shear deformation and rotary inertia on the natural frequencies of pre-twisted cantilever beams using numerical technique. They also verified their results conducting experiments. Carnegie and Thomas[4] used finite difference method to study the vibration of uniform and tapered pre-twisted cantilevers. Subrahmanyam and Rao[5] applied Reissner method to study the vibration of tapered pre-twisted cantilever beam. Onipede and Dong[6] studied vibration of pre-twisted inhomogeneous beam of arbitrary cross-section by using variational method. Vibration and stability of a spinning pre-twisted thin walled composite beam were studied by Song, Jeong and Librescu[7] considering a number of non-classical features such as transverse shear, anisotropy and pre-twist. Vielsack[8] has shown that the influence of small pre-twist on lateral vibration of beams depends on the ratio of bending stiffness about principal axes.

Lin, Wu and Lee [9] have studied the coupled bending–bending vibration of a rotating pre-twisted beam with an elastically restrained root and a tip mass, subjected to the external transverse forces and rotating at a constant angular velocity. Yardimoglu and Yildirim[10] have developed a finite element model with reduced number of nodal degree of freedom to investigate the vibration of a pre-twisted cantilever beam. The effect of pre-twist angle of an aerofoil blade simplified as a rotating Euler as well as Timoshenko beam has been investigated by Subuncu and Ervan[11,12] using finite element method. Jhung and Jo [13] have studied the vibration characteristics of a rectangular twisted beam with pins surrounded with liquid. Mohanty[14] has studied parametric instability of pre twisted cantilever with localized damage. Hsu [15] has investigated dynamic behaviour of pre-twisted beams using spline collocation method. Liu, Friend and Yeo[16] have carried out an investigation on the coupled axial-torsional vibration of pre-twisted beams. Leung and Fan [17] have studied the influence of multiple kinds of initial stresses due to compression, shears, moments and torque on the natural vibration of pre-twisted straight beam based on the Timoshenko theory. Chen [18, 19] has found the influence of thickness- to-width ratio, twist angle, spinning speed and axial load on the natural frequency, buckling load and instability zone of a pre-twisted Timoshenko beams by using finite element method. Mohanty, Dash and Rout [20,21] studied static and dynamic behaviour of Timoshenko beam on Winkler’s foundation using finite element method.

FGM made of the mixture of turbine material and ceramics can be applied as a coating on the surface of the blade such that the surface of the coated blade is rich in ceramics to improve the performance of the blade in eroding environment. This article presents a study of free vibration of a functionally graded pre-twisted Timoshenko beam of cantilever type.

II. FORMULATION

A functionally graded pre-twisted beam is considered for analysis as shown in Fig.1 (a). The mid-longitudinal(x-y, x-z) planes are chosen as the reference planes for expressing the displacements as shown in fig. 1(b). The two transverse displacement of a point on the reference plane are, v and w and the corresponding rotation of cross-sections are $\phi$ and $\theta$.

Fig. 1(c) shows a two nodded beam finite element having four degrees of freedom per node.
A. Shape functions
The differential equation of motion [2, 3, 4] for the FGSW beam element is given as
\[ \frac{\partial M_y}{\partial x} + V_z = -\left(\rho_1 \cos^2 \alpha(x) + \rho_2 \sin^2 \alpha(x)\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{2} \left(\rho_2 - \rho_1\right) \sin 2\alpha(x) \frac{\partial^2 \theta}{\partial x^2} \] (1)
\[ \frac{\partial M_z}{\partial x} + V_y = \left(\rho_2 \cos^2 \alpha(x) + \rho_1 \sin^2 \alpha(x)\right) \frac{\partial^2 \phi}{\partial x^2} - \frac{1}{2} \left(\rho_2 - \rho_1\right) \sin 2\alpha(x) \frac{\partial^2 \theta}{\partial x^2} \] (2)

\[ \frac{\partial V_y}{\partial x} = \rho_0 \frac{\partial^2 w}{\partial t^2} \]
\[ \frac{\partial V_z}{\partial x} = \rho_0 \frac{\partial^2 w}{\partial t^2} \]
where
\[ M_y = -I_1 \cos^2 \alpha(x) + I_2 \sin^2 \alpha(x) \frac{\partial \phi}{\partial x} - \frac{1}{2} \left(\rho_2 - \rho_1\right) \sin 2\alpha(x) \frac{\partial \theta}{\partial x} \]
\[ M_z = \left(\rho_2 \cos^2 \alpha(x) + \rho_1 \sin^2 \alpha(x)\right) \frac{\partial \phi}{\partial x} + \frac{1}{2} \left(\rho_2 - \rho_1\right) \sin 2\alpha(x) \frac{\partial \theta}{\partial x} \]
\[ V_y = -G_y \left(\frac{\partial w}{\partial x} - \phi\right) \]
\[ V_z = G_z \left(\frac{\partial w}{\partial x} - \theta\right) \]
\[ I_1 = \int E(z') \left(z'^2\right) dA, \quad I_2 = I_{1b} \frac{b^2}{12}, \quad I_0 = \int E(z') dA, \]
\[ \rho_2 = \frac{\rho_0 b^2}{12}, \quad \rho_1 = \rho_0 \int \rho(z') dA, \quad G_0 = k \int G(z') dA. \]
\[ \alpha(x) = \alpha_0 + \xi x \]
\[ R(z') = (R_i - R_s) \left(\frac{z'}{h} + \frac{1}{2}\right) + R_s \] (6)
where, \( R(z') \) denotes a material property such as, \( E, G, \rho \)
etc., \( R_i \) and \( R_s \) denote the values of the properties at topmost and bottommost layer of the beam respectively, and \( n \) is an index.

Eq.(1,2,3,4) in homogeneous form is given as
The following polynomials may be assumed for the displacement field.

\[
\begin{align*}
  v &= a_1 + a_2 x + a_3 x^2 + a_4 x^3, \\
  w &= b_1 + b_2 x + b_3 x^2, \\
  \phi &= c_1 + c_2 x + c_3 x^2 + c_4 x^3, \\
  \theta &= d_1 + d_2 x + d_3 x^2 + d_4 x^3,
\end{align*}
\]  

\begin{equation}
(11)
\end{equation}

where \( \{ \mathbf{u} \} \) and \( \{ \mathbf{r} \} \) are deflection and rotation vector respectively.

\[
N_v, N_w, N_\theta \text{ and } N_\phi \text{ are the corresponding shape functions.}
\]

The coefficient vector \( \mathbf{b} \) can be expressed in terms of the coefficient vector \( \mathbf{a} \) by substituting eq. (11) in eq. (7) and by equating the coefficients of same powers of \( x \). The mentioned procedure yields the following.

\[
\begin{align*}
  b_1 &= \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 + \beta_4 a_4, \\
  b_2 &= \beta_1 a_2 + \beta_3 a_3 + \beta_4 a_4, \\
  b_3 &= a_2 + \beta_3 a_3 + \beta_4 a_4, \\
  b_4 &= 2a_2 + \beta_4 a_4.
\end{align*}
\]

\begin{equation}
(14)
\end{equation}

where, \( \beta_1 = 2\phi_1 \cos 2\alpha(x) \), \( \beta_2 = 6\phi_1 \cos 2\alpha(x) \phi_1 \sin 2\alpha(x) \), \( \beta_3 = 6\phi_1 \cos 2\alpha(x) \), \( \beta_4 = 6\phi_1 \sin 2\alpha(x) \), \( \beta_5 = -\beta_3 \), \( \beta_6 = \beta_1 \), \( \beta_7 = -\beta_6 \), \( \beta_8 = I_0 \)

\[\beta_9 = (I_0 - I_0)/G_0 \]

The coefficient vector \( \mathbf{b} \) can be expressed in terms of the coefficient vector \( \mathbf{a} \) in matrix form as

\[
\{ \mathbf{b} \} = [\mathbf{B}] \{ \mathbf{a} \}
\]

\begin{equation}
(15)
\end{equation}

A two nodded Timoshenko beam finite element with 4-degree of freedom per node as shown in fig. 1(c) is considered for analysis. The nodal degree of freedom vector of the \( i \)th element is given as

\[
\{ \mathbf{u}_i \} = [v_i \ w_i \ \phi_i \ \theta_i \ v_{i+1} \ w_{i+1} \ \phi_{i+1} \ \theta_{i+1}]
\]

\begin{equation}
(16)
\end{equation}

Using eq. (11) and eq. (14) the nodal degree of freedom vector can be expressed as

\[
\{ \mathbf{u} \} = [C][\mathbf{a}]
\]

\begin{equation}
(17)
\end{equation}

\[
\begin{bmatrix}
0 & 0 & \beta_1 & \beta_2 & 0 & 1 & \beta_3 & \beta_4 \\
0 & 0 & 0 & \beta_4 & 0 & 2 & \beta_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 1 & \beta_7 & \beta_8 & 0 & 0 & \beta_3 & \beta_2 \\
0 & 0 & 2 & \beta_3 & 0 & 0 & \beta_1 & \beta_2 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Substituting eq. (15) in eq. (13) we get

\[
\{ \phi \} = [C_a][\mathbf{u}]
\]

\begin{equation}
(18)
\end{equation}

\[
\{ \theta \} = [C_\theta][\mathbf{u}]
\]

\begin{equation}
(19)
\end{equation}

B. Element elastic stiffness matrix

The strain energy of the element [4] is given as

\[
S = \frac{1}{2} \int_0^L \left( \int \left[ \frac{\partial^2}{\partial x^2} \right] \right) \left[ \frac{\partial^2}{\partial x^2} \right] + \left[ \frac{\partial^2}{\partial x^2} \right] \sin 2\alpha(x) \right) \right) \]  

\begin{equation}
(20)
\end{equation}

Substituting eq. (5, 12 18 and 19) into eq. (20) the expression of strain energy is given as

\[
S = 0.5[\mathbf{u}]^T \{ k \} [\mathbf{u}]
\]

\begin{equation}
(21)
\end{equation}

where \( \{ k \} \) is the element elastic stiffness matrix and is given as follows.

\[
\{ k \} = \int_0^L \left[ \frac{\partial^2}{\partial x^2} \right] \left[ \frac{\partial^2}{\partial x^2} \right] \]  

\begin{equation}
(22)
\end{equation}

C. Element mass matrix

The kinetic energy of the beam element [4] is expressed as follows

\[
T = \frac{1}{2} \int_0^L \left[ \int \left[ \frac{\partial^2}{\partial x^2} \right] \left[ \frac{\partial^2}{\partial x^2} \right] + \left[ \frac{\partial^2}{\partial x^2} \right] \sin 2\alpha(x) \right) \right) \]  

\begin{equation}
(23)
\end{equation}

Substituting eq. (5, 12 18 and 19) in eq. (23) the kinetic energy is expressed as follows.

\[
T = 0.5[\mathbf{u}]^T \{ m \} [\mathbf{u}]
\]

\begin{equation}
(24)
\end{equation}

where \( \{ m \} \) is the element mass matrix which is presented below

\[
\{ m \} = \int_0^L \{ C \}^T \{ D \} [C] \]  

\begin{equation}
(25)
\end{equation}
D. Governing equation of motion

The equation of motion for the element can be expressed in terms of nodal degrees of freedom as

\[
[D] = \rho_s \left[ [N_u] [N_u] + [N_v] [N_v] \right] + \left[ \rho_s \cos^2 \alpha(x) + \rho_s \sin^2 \alpha(x) \right] [N_u] [N_u] + \left[ \rho_s \cos^2 \alpha(x) + \rho_s \sin^2 \alpha(x) \right] [N_v] [N_v] + [B]
\]

Assembling the element matrices as used in eq. (40), the equation in global matrix form which is the equation of motion for the beam, can be expressed as

\[
[M] \ddot{\mathbf{u}} + [K] \mathbf{u} = 0
\]

\[
[M], [K], \quad \text{are global mass, elastic stiffness matrices respectively and } [\mathbf{u}] \quad \text{is global displacement vector.}
\]

III. RESULTS AND DISCUSSION

A steel-alumina FGO beam with steel-rich bottom, fixed at one and free at other is considered for vibration and dynamic analysis. The length, breadth and thickness of the beam are 50.8 cm, 2.54 cm and 0.03175 cm respectively. The material properties of the constituent phases are given below. Properties of steel: \( E = 2.1 \times 10^{11} \text{ Pa}, \quad G = \frac{3}{8}kE \quad \rho = 7.8 \times 10^3 \text{kg/m}^3 \). Properties of alumina: \( E = 3.9 \times 10^{11} \text{ Pa}, \quad G = \frac{3}{8}kE \), \( \rho = 2.707 \times 10^3 \text{kg/m}^3 \). The shear correction factor is assumed as \( k = 0.833 \).

The effect of power law index on the natural frequencies of the FGO beam is investigated and is presented in fig. 2(a) and 2(b) for first and second mode respectively. It is observed from the figure that the increase in power law index decreases the first and second mode frequencies. This may be due to the fact that increase in index causes increase in the relative amount of steel which in turn decreases the elastic stiffness of the beam.
The Fig. 3(a) and 3(b) represent the variation of first and second mode frequencies respectively with the pre-twist angle. The increase in pre-twist angle increases the first mode frequency of the beam where as it decreases the second mode frequency of the beam.

IV. CONCLUSION

Finite element method in conjunction with static equilibrium approach is used to investigate the free vibration behaviour of a steel-alumina functionally graded cantilever beam with steel-rich bottom. The material properties along the thickness of the beam are assumed to vary according to power law. Increase in the value of power law index decreases the first two modal frequencies of the beam. Increase in pretwist angle increases first mode frequency and decreases the second mode frequency.

Nomenclature:

\{a\} Independent coefficient vector
\{b\} Dependent coefficient vector
\{a_1, a_2, a_3, a_4\} Polynomial coefficients of the linear
displacement on xy plane
\{b_1, b_2, b_3, b_4\} Polynomial coefficients of the linear displacement on
xz plane
\{I_0, I_1, I_2\} Stiffness coefficients
\{\rho_0, \rho_1, \rho_2\} Mass moments
\{M_y, M_z\} Bending moments about Y and Z axis
\{V_y, V_z\} Shear forces along Y and Z axis
\{R_b\} Material property at the bottommost layer
\{R_t\} Material property at topmost layer
\{\rho(z)\} A material property at location z'
\{\eta\} Natural frequency
\{\phi\} Rotation of cross-section plane about Y- axis
\{\theta\} Rotation of cross-section plane about Z- axis
\{\alpha_0\} Twist angle at the free end
\{\varphi\} Increase in twist angle per unit length
\{\bar{\mu}_d, \bar{\mu}_r\} Deflection and Rotation vectors
\{\rho(z)\} Density of material at location z'
\{K_e\} Global elastic stiffness and mass matrices
\{N_x, N_w, N_y, N_\theta\} Shape function matrix
\{\hat{u}\} Nodal displacement vector
\{\hat{u}'\} Global nodal displacement vector
\{k_e\} Element elastic stiffness and mass matrices
\{\hat{G}\} Material constant matrix

REFERENCE

Optimisation of EDM Process with Fuzzy Logic Technique

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\textbf{Abstract}-- In this analysis, the optimisation of multiple responses of Electric discharge machining (EDM) using Fuzzy logic coupled with Taguchi method is attempted. The work piece material was AISI P20 tool steel and a cylindrical copper electrode was used with side impulse flushing. The influence of machining parameters, i.e., pulse current (Ip), pulse duration (Ton), work time (Tw), lift time (Tup) and Inter Electrode Gap (IEG) on the Material Removal Rate (MRR) and Surface Roughness(SR) in EDM are examined. L27 orthogonal array was used to design the experiment and the effect of the factors on the responses were studied. Experimental data has been analysed using analysis of variance (ANOVA). As the responses are conflicting in nature, a single combination of factors cannot be treated as best machining performance for all responses. Fuzzy logic is used to convert multiple responses into a single characteristic index known as Multi Performance Characteristic Index (MPCI). Finally, MPCIs were optimised by using robust Taguchi design.

\textbf{Keyword}-- Electric Discharge Machining (EDM), Fuzzy logic, Impulse flushing, AISI P20 tool steel.

\section{Introduction}

EDM is an electro-thermal non-traditional machining Process, where electrical energy is used to generate electrical spark and material removal mainly occurs due to thermal energy of the spark. EDM is mainly used to machine difficult-to-machine materials and high strength temperature resistant alloys. It is the most widely and successfully applied machining process for various work piece materials in the said advance industries \cite{1}. With the advancement of automation technology manufacturers are more fascinated in the processing and miniaturization of components made by these costly and hard materials. Electrical discharge machining (EDM) has grown over the last few decades from a novelty to a mainstream manufacturing process. Work material to be machined by EDM has to be electrically conductive. Quite a lot of research attempts have been made for modelling of EDM process and investigation of the process performance to recuperate MRR. Improving the MRR and surface quality are still challenging problems that restrict the expanded application of the technology \cite{2}.

A semi-empirical model of MRR for various work piece and tool electrode combinations has been presented by Wang and Tsai \cite{3}. Luis et al. \cite{4} have studied the influence of pulse current, pulse time, duty cycle, open-circuit voltage and dielectric flushing pressure, over the MRR and other response variable on tungsten carbide. In this paper, optimisation of multiple performance characteristics to single optimization characteristics with help of fuzzy logic system has been attempted.

\section{Conduct of Experiment}

Experiments were conducted to study the effects of various machining parameters on EDM process. This study have been undertaken to investigate the effects of Discharge current (Ip), pulse on time (Ton) and work time (Tw), lift time (Tup) and Inter Electrode gap (IEG) on Material Removal Rate (MRR) and Surface Roughness (SR). The selected work piece material is AISI P20 tool steel with semi-circular shaped (100 mm diameter and 10 mm thickness). The workpiece was heated to the temperature range $843 - 898$ °C in a controlled furnace and held for half an hour. Then it was oil quenched and later tempered for better toughness. A cylindrical pure copper with a diameter of 30 mm was used as a tool. The experiments were conducted on Electronica Electraplus PS 50ZNC Die Sinking Machine. Commercial grade EDM oil (specific gravity = 0.763, freezing point= 94°C) was used as dielectric fluid with impulse flushing pressure of 0.3 kgf/cm$^2$.

The machining parameter and their level are presented in Table 1. The five machining parameters, each at three levels, were selected based on Orthogonal Array (OA), resulting in Twenty-seven experiments. The observations of the experiments are tabulated in Table 2. To attain more accurate results, every combination runs of experiments was machined for 60 min and then MRR was calculated by weight loss method. Central line average (Ra) value of the machined surface was measured the surface roughness, it usually expressed as Ra value in microns. Portable style type profile-meter, Talysurf (Model: Taylor Hobson, Surtronic 3+) was used for measurement with parameters cut-off length, Ln=4mm, sample length, Lc= 0.8mm and filter=2CR ISO.

\section{Experimentation}

In this experiment, lower value of SR is desired quality and higher for MRR. Therefore, SR is the lower-the-better performance characteristic and the MRR is the higher-the-better performance characteristic. MRR and SR values are converted to S/N ratio. S/N ratio $x_{ij}$ is the $i^{th}$ performance