Thermo-electro-elastic Analysis of Transversely Isotropic Piezoelectric Long Solid Cylinder under Thermal Loading

Mohammad H. Kargarnovin and Mehdi Hashemi

Abstract—In this paper, an analytical solution is presented for thermo-electro-elastic analysis of piezoelectric long solid cylinder subjected to thermal loading under axisymmetric conditions. Governing equations for transversely isotropic piezoelectric material accompanied by temperature distribution field are presented in the cylindrical coordinate system. The complementary and particular solutions for components of displacements and electric potential function are proposed in terms of cosine and sinuous Fourier integral transforms by solving a set of three non-homogenous governing partial differential equations, as well as the temperature distribution function for this axisymmetric piezoelectric cylinder. Thermal loading resulted from heat convection of flowing fluid is exerted on the surface, and then in an illustrative example, variations of stress component and electric field in radial direction are studied when fluid temperature distribution is in the form of step function.

Keywords—Analytical Solution, Piezoelectric, Thermo-electro-elastic Analysis, Axisymmetric Solid Cylinder.

I. INTRODUCTION

In recent decades, employing piezoelectric materials including crystal or ceramic types in the growing field of intelligent structures and control systems as actuators and sensors due to their distinct features has been dramatically increased. So studying thermo-electro-elastic coupling behavior in these devices is of significant importance.

Among all geometry shapes made out of piezoelectric materials, solid cylinder are widely used. Researches carried out for the analysis of these cylinders can be classified into two categories, radially and axially polarized ones. With the respect to radially polarized cylinders, vibrations of them were firstly investigated by Adelmann and Stavsky [1, 2]. Ding and Wang [3] developed a method for solving the transient response of these cylinders subjected to dynamic loads by proposing analytical solutions for components of displacements and electrical potential function. Then, in a numerical example, variations of stress component and electric field in radial direction are studied when the piezoelectric subjected to folwing of a fluid.

II. GOVERNING EQUATIONS

Suppose a transversely isotropic piezoelectric long solid cylinder with the axis of polarization collinear with the z-axis under axisymmetric conditions subjected to a fluid flow in a cylindrical coordinate system (r, θ, z) shown in Fig. 1. If $\sigma_{ij}$, $e_{ij}$, $D_{ij}$, $E_{ij}$, $C_{ij}$, $e_{ij}$ are components of stress tensor, strain tensor, electric displacement vector, electric field vector, elastic constants, piezoelectric constants, respectively, then the constitutive equations presented by references [8, 9] with additional terms to include the temperature distribution effects can be expressed as:

$$\sigma_{rr} = c_{11} e_{rr} + c_{12} e_{\theta\theta} + c_{13} e_{zz} - e_{31} E_z - \beta z; \quad (1)$$

$$\sigma_{\theta\theta} = c_{12} e_{rr} + c_{11} e_{\theta\theta} + c_{13} e_{zz} - e_{31} E_z - \beta z; \quad (2)$$

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\[ \sigma_{zz} = c_{11} \varepsilon_{rr} + c_{12} \varepsilon_{\theta \theta} + c_{13} \varepsilon_{zz} - \beta_3 \tau; \]  
\[ \sigma_{rz} = 2c_{44} \varepsilon_{rz} - e_{33} E_z; \]  
\[ D_r = 2e_{15} \varepsilon_{rr} + e_{11} E_r; \]  
\[ D_z = e_{31} \varepsilon_{rr} + e_{32} \varepsilon_{\theta \theta} + e_{33} \varepsilon_{zz} - \beta_4 \tau, \]

\[
\frac{\partial T}{\partial r} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r} \frac{\partial^2 T}{\partial z^2} = 0; \\
\text{and } \tau = T(r, z) - T_f(z),
\]

\[
\beta_1 = \beta_2 = c_{11} \alpha_r + c_{12} \alpha_\theta + c_{13} \alpha_z; \\
\beta_3 = c_{11} \alpha_r + c_{13} \alpha_\theta + c_{15} \alpha_z; \\
\beta_4 = c_{31} \alpha_r + c_{32} \alpha_\theta + c_{33} \alpha_z,
\]

In which \( \alpha_i \) \( (i=r, \theta, z) \) are thermal expansion coefficients. Since \( \alpha_r = \alpha_\theta \), this yields that \( \beta_1 = \beta_2 \).

The equilibrium equations and Gauss’ equation representing the balance of electric displacements for a piezoelectric cylinder in axisymmetric conditions can be written as [8, 9]:

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{\theta \theta}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta \theta}}{r} = 0, \\
\frac{\partial \sigma_{\theta \theta}}{\partial r} + \frac{\partial \sigma_{rr}}{\partial z} + \frac{\sigma_{\theta \theta} - \sigma_{rr}}{r} = 0, \\
\frac{\partial D_r}{\partial r} + \frac{\partial D_\theta}{\partial z} + \frac{D_r}{r} = 0.
\]

Furthermore, the strain-displacement relations and the relations between the electric field components, \( E_i \), and electric potential \( \phi \) can be given by [8, 9]:

\[
E_r = \frac{\partial \varphi}{\partial r}; \\
E_{\theta \theta} = \frac{u}{r}; \\
E_z = \frac{\partial \varphi}{\partial z}; \\
E_r = 1 \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right),
\]

\[
E_z = -\frac{\partial \varphi}{\partial z},
\]

where \( u \) and \( w \) denote displacement components in \( r \) and \( z \) directions, respectively. Combinations of (1)-(6) and (9)-(13) result in the following a set of non-homogenous differential equations:

\[
\frac{\partial \varphi}{\partial r} = \frac{c_{11}}{k_r} \left( \frac{\partial \varphi}{\partial r} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{c_{13}}{e_{11}} \frac{\partial \varphi}{\partial z^2} \right) + \frac{c_{44}}{e_{11}} \left( \frac{\partial \varphi}{\partial r} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{c_{13}}{e_{11}} \frac{\partial \varphi}{\partial z^2} \right),
\]

\[
\frac{\partial \varphi}{\partial z} = \frac{c_{11}}{k_z} \left( \frac{\partial \varphi}{\partial r} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{c_{13}}{e_{11}} \frac{\partial \varphi}{\partial z^2} \right) + \frac{c_{44}}{e_{11}} \left( \frac{\partial \varphi}{\partial r} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{c_{13}}{e_{11}} \frac{\partial \varphi}{\partial z^2} \right).
\]

### III. Analytical Solutions

#### A. General solutions

The analytical general solution for the temperature change distribution function, (7), in a long transversely isotropic long solid cylinder can be expressed as:

\[
\tau(r, z) = A_0 + \int_0^s A(s) I_0 \left( \frac{k_s}{k_r} \right) \cos(sz) ds - T_f(r, z),
\]

In which \( A_0 \) and \( A(s) \) are unknown functions which will be determined from boundary conditions, and \( I_0 \) denotes the modified Bessel function of the first kind and zero order.

Furthermore, the analytical general solution for a set of non-homogenous partially differential equations of (14) for a long piezoelectric solid cylinder consists of two terms, one for non-homogenous part of equations as particular solution and one for homogenous part as complimentary one. The complimentary solution itself consists of summation of three terms as:

\[
\frac{\partial \varphi}{\partial r} = \frac{c_{11}}{k_r} \left( \frac{\partial \varphi}{\partial r} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{c_{13}}{e_{11}} \frac{\partial \varphi}{\partial z^2} \right) + \frac{c_{44}}{e_{11}} \left( \frac{\partial \varphi}{\partial r} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{c_{13}}{e_{11}} \frac{\partial \varphi}{\partial z^2} \right).
\]
\( u = \int_{0}^{\infty} \left( B(s)I_{0}(s \sqrt{\frac{k}{k'}}) + \sum_{j=1}^{3} \delta_j H_j(s)I_{0}(sk',r) \right) \cos(sz)ds, \) \hspace{1cm} (16)

\( w = \int_{0}^{\infty} \left( C(s)I_{0}(s \sqrt{\frac{k}{k'}}) + \sum_{j=1}^{3} \eta_j H_j(s)I_{0}(sk',r) \right) \sin(sz)ds, \) \hspace{1cm} (17)

\( \varphi = \int_{0}^{\infty} \left( D(s)I_{0}(s \sqrt{\frac{k}{k'}}) + \sum_{j=1}^{3} \xi_j H_j(s)I_{0}(sk',r) \right) \sin(sz)ds, \) \hspace{1cm} (18)

Where \( B(s), C(s) \) and \( D(s) \) functions can be obtained from (19), \( \delta_j, \eta_j \) and \( \xi_j \) from (20) \( H_j \) from boundary conditions, and finally \( k_j \) can be computed by putting determinant of known matrix \( P \) equals to zero. It should be noted that the resulted bicubic equation in general has three pairs of roots, \((\pm k_1, \pm k_2, \pm k_3)\) where \( k_1 \) is a positive real number and \( k_2 \) and \( k_3 \) are either positive real numbers or a pair of complex conjugates with positive real parts.

\( \{B(s), C(s), D(s)\}^T = [M]^{-1}[N], \) \hspace{1cm} (19)

in which

\[
[M] = \begin{bmatrix}
(c_{13} + c_{44})k' \frac{k}{k'} - c_{44} & (c_{13} + c_{44})k' \frac{k}{k'} & (e_{31} + e_{15})k' \frac{k}{k'} \\
(c_{13} + c_{44})k_2 - c_{44}k_2 & (c_{13} + c_{44})k_2 - c_{44}k_2 & (e_{31} + e_{15})k_2 - c_{44}k_2 \\
(e_{31} + e_{15})k_3 - c_{44}k_3 & (e_{31} + e_{15})k_3 - c_{44}k_3 & (e_{31} + e_{15})k_3 - c_{44}k_3 \\
\end{bmatrix}
\]

\[
[N] = \begin{bmatrix}
\beta_1 k' \frac{k}{k'} \\
\beta_2 k' \frac{k}{k'} \\
\beta_3 k' \frac{k}{k'}
\end{bmatrix}
\]

Thus, by taking derivatives from displacement vectors and electric potential functions, components of stress and electrical displacement can be easily computed from (1)-(6).

### B. Imposing Boundary Conditions

The boundary condition considered for this long transversely isotropic piezoelectric solid cylinder is purely thermal in form of a fluid flow as:

\( T = T_i \) at \( t = 0; \) \( T = T_f \) at \( z \to \infty; \)

\( -k_r \frac{\partial T}{\partial r} = h_f(T - T_f(z)), \) on \( r = a, \)

\( D_r = 0; \) \( \sigma_r = 0; \) \( \sigma_r = 0 \) on \( r = a, \) \hspace{1cm} (22)

in which \( T(z) \) is fluid temperature distribution. Thus, from (22) \( A_0 \) and \( A(s) \) can be easily found as:

\[ A_0 = T_i; \] \[ A(s) = h_f(s) \]

\[
\frac{k_j}{k_j} \frac{k}{k'} + s \sqrt{k_j k'} I_s \left( sa \frac{k}{k'} \right) \]

where

\[ \tilde{f}(s) = 2 \pi \int_{0}^{\infty} [T(z) - T_i] \cos(sz)dz, \]

And from (23) \( H_j(j=1, 2 \text{ and } 3) \) can be determined as:

\[ \{H_1(s), H_2(s), H_3(s)\}^T = [X]^{-1}[Y], \] \hspace{1cm} (25)

where the elements of matrix \( [X] \) and vector \( \{Y\} \) are given as follow:
To illustrate the effect of a fluid flow on thermo-electro-elastic behavior of long transversely isotropic piezoelectric solid cylinder, in this example, we assumed that the fluid temperature distribution is $T_f(z) = T_i + T_a H(b-|z|)$ in which $T_i$ and $T_a$ are constants and $H(b-|z|)$ is Heaviside or step function. For numerical analysis; $h=10$ w/(m²°C) and $a=b=1$ m are assumed. Meanwhile, the piezoelectric material considered for this cylinder is PZT-6B whose properties are given in Table I.

### TABLE I

<table>
<thead>
<tr>
<th>Material properties of PZT-6B [9]</th>
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<tbody>
<tr>
<td>Elastic coefficients ($10^{10}$ N m⁻²)</td>
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<td>$c_{11}$</td>
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<td>$c_{12}$</td>
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<td>$c_{33}$</td>
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<td>$c_{44}$</td>
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<td>$c_{66}$</td>
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<th>Piezoelectric coefficients (C/m²)</th>
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<td>$e_{33}$</td>
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<td>$e_{31}$</td>
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<td>$e_{15}$</td>
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<th>Thermal conductivity (W/m°C)</th>
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<tr>
<td>$k_r$</td>
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<td>$k_z$</td>
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The values of $k_i$ obtained for this cylinder are $k_i=0.475866$, $k_1=0.985855$ and $k_t=1.93200$.

In Figs. 2 and 3, variations of dimensionless radial stress and electric field vs. axial length are illustrated at different radial length. Note that normalizing factors for $\sigma_r$ and $E_r$ are $\beta_{1T_a}$ and $\beta_{2T_a}/e_{15}$, respectively. As it is observed in these figures, the value of radial stress increases as one approaches near to the center, while the value of radial electric field decreases as one moves towards the center. The maximum value of radial stress occurs at different axial length for each radial length, while the maximum value of radial electric field occurs at $z/a=1$ where no temperature input exists afterwards for all radial length.

### IV. AN ILLUSTRATIVE EXAMPLE

To illustrate the effect of a fluid flow on thermo-electro-elastic behavior of long transversely isotropic piezoelectric solid cylinder, in this example, we assumed that the fluid temperature distribution is $T_f(z) = T_i + T_a H(b-|z|)$ in which $T_i$ and $T_a$ are constants and $H(b-|z|)$ is Heaviside or step function. For numerical analysis; $h=10$ w/(m²°C) and $a=b=1$ m are assumed. Meanwhile, the piezoelectric material considered for this cylinder is PZT-6B whose properties are given in Table I.

### V. CONCLUSION

In this paper, an analytical solution is presented for thermo-electro-elastic analysis of piezoelectric long solid cylinder polarized in the axial direction under thermal loading. The analysis was performed when the cylinder subjected to the flowing of a fluid. Based on partial distribution considered for the fluid temperature in the numerical example, it is concluded that the value of radial stress increases as one approaches near to the center, while the value of radial electric field decreases...
as one moves towards the center. The maximum value of radial stress occurs at different axial length for each radial length, while the maximum value of radial electric field occurs where no temperature input exists afterwards for all radial length.

REFERENCES


