Finite State Automaton Group

K. Thiagarajan¹, S. Jeya Bharathi², A. Jeyanthi³, and J. Padmashree⁴

Abstract—This paper deals with the Group structure of Finite State Machine when δ is total. Also we defined the sub FSM group, Normal FSM group and the partially ordered FSM group. Finite state Machine may be taken into account for the transfer of topological structure in different way, to intend different applications of finite automata to group Theory, mentioning also some generalizations to the wider context of monoids.

Keywords— FSM Group, Sub FSM Group, Normal FSM Group, J-trivial Group, Partially ordered FSM.

I. INTRODUCTION

Previous algebraic investigations of Finite-State Machines (FSM) have made use of a semi group structure defined on finite strings of elements taken from the input set of the machine. Representations for the machines [1] are then given in terms of certain homomorphic mappings of these input strings [2]. We give the group structure for the finite state machines in a complete FSM. (ie) M→M, the modulo operation defined by γ (δ(q₁, x₁), δ(q₂, x₂), ..., δ(qₙ, xₙ)) = δ(q₁, x₁) ⊕ δ(q₂, x₂) ⊕ ... ⊕ δ(qₙ, xₙ)

DEFINITION 2.1: FSM GROUP
A non-empty set M where (M = {δ(q₁, x₁) / q₁ ∈ Q, x₁ ∈ Σ, δ ∈ δ(q₁, x₁)}) is said to be a FSM Group with map γ : M x M→M, the modulo operation defined by γ(δ(q₁, x₁), δ(q₂, x₂), ..., δ(qₙ, xₙ)) = δ(q₁, x₁) ⊕ δ(q₂, x₂) ⊕ ... ⊕ δ(qₙ, xₙ) where n represents number of states in the machine, and m represents number of inputs in the machine.

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We have to check $M$ is a group under the operation defined by
\[
\delta(t_i, x_k) \oplus \delta(t_j, x_l) = \delta(t_i \oplus t_j, x_k \oplus x_l)
\]
where $i, j, k, l$ are non-negative integers and $|Q| = n, |\Sigma| = m$.

**Closure**
\[
\gamma(\delta(t_0, x_0), \delta(t_0, x_1)) = \delta(t_0, x_1) \in M
\]
\[
\gamma(\delta(t_0, x_0), \delta(t_1, x_0)) = \delta(t_1, x_0) \in M
\]
\[
\delta(t_0, x_0) \gamma(t_1, x_1) = \delta(t_1, x_1) \in M
\]
\[
\delta(t_0, x_0) \gamma(t_2, x_1) = \delta(t_2, x_1) \in M
\]
\[
\delta(t_0, x_0) \gamma(t_1, x_1) = \delta(t_0, x_1) \in M
\]

**Associative**
\[
\gamma(\delta(t_1, x_2), \delta(t_2, x_1)) = \delta(t_2, x_2)
\]
\[
\gamma \oplus \delta(t_2, x_1) = \delta(t_3, x_1)
\]
\[
\gamma(\delta(t_1, x_2), \delta(t_2, x_1)) = \delta(t_0, x_2)
\]
\[
\gamma \oplus \delta(t_3, x_1) = \delta(t_0, x_1)
\]

From (1) and (2),
\[
\gamma(\delta(t_1, x_2), \delta(t_2, x_1)) = \gamma(\delta(t_0, x_2), \delta(t_3, x_1))
\]
\[
\gamma \oplus \delta(t_3, x_1) = \delta(t_0, x_1)
\]

**Inverse Element**
There exist an element $\delta(t_i^{-1}, x_i^{-1}) \in M$ such that
\[
\gamma(\delta(t_i, x_i), \delta(t_i^{-1}, x_i^{-1})) = \delta(t_i, x_i)
\]
\[
\gamma \delta(t_i^{-1}, x_i^{-1}) = \gamma \delta(t_i, x_i)
\]
where $i, k$ are non-negative integers.

Hence $M$ is FSMGroup.

**Definition 2.2**
A group $M$ is said to be abelian (or commutative) if for every $\delta(t_j, x_j), \delta(t_i, x_i) \in GM$
\[
\gamma(\delta(t_j, x_j), \delta(t_i, x_i)) = \gamma(\delta(t_i, x_i), \delta(t_j, x_j))
\]

**Result 2.1**
Another natural characteristic of a group $GM$ is the number of elements it contains. We call this the order of $M$ and denoted by $O(M)$. This number is finite we say that $M$ is a finite group.

**Result 2.2**
From Result 1, we conclude that, in a finite state machine, $O(M)$ represents the number of possible path between one state to another [5] state.

**Result 2.3**
A finite state machine $M = (Q, \Sigma, \Delta, \delta, \lambda)$ the cardinality of $|Q| = n, |\Sigma| = m$, then the cardinality of $G_M$
\[
\text{(ie) } M = O(M) = n \times m.
\]

**Example 2.1**
A finite state machine which is not a FSM Group.

<table>
<thead>
<tr>
<th>State</th>
<th>Input $x_0$</th>
<th>Input $x_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_0$</td>
<td>$q_0$</td>
<td>$q_1$</td>
</tr>
<tr>
<td>$q_1$</td>
<td>$q_0$</td>
<td>$q_2$</td>
</tr>
<tr>
<td>$q_2$</td>
<td>$q_0$</td>
<td>$q_1$</td>
</tr>
</tbody>
</table>

**TABLE II**

<table>
<thead>
<tr>
<th>State</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_0$</td>
<td>$q_0$</td>
</tr>
<tr>
<td>$q_1$</td>
<td>$q_0$</td>
</tr>
<tr>
<td>$q_2$</td>
<td>$q_0$</td>
</tr>
</tbody>
</table>
Let the given conditions hold in $H_M$. We show $H_M$ forms a subgroup of $G$ iff (i) It is closed under the binary operation in $G$. (ii) The identity $\gamma(\delta(q_0,x_0), \delta(q_1,x_1)) \in H_M$. (iii) $\delta(s_0,x_0), \delta(s_1,x_1) \in H_M$.

**Definition 2.3 sub FSM Group**

A non-empty subset $H_M$ of a group $M$ is said to be a sub FSM Group of $M$, if $H_M$ forms a Group under modulo operation of $M$.

Consider a finite state machine $M=(Q, \Sigma, \Delta, \delta, \lambda)$ with $Q=\{q_0, q_1, q_2\}$, $\Sigma=\{\sigma_0, \sigma_1\}$ and the next–state function $\delta$ defined by Table II whose State diagram is given below Refer figure figure II

![State Diagram](image)

<table>
<thead>
<tr>
<th>State</th>
<th>Input $\sigma_0$</th>
<th>Input $\sigma_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_0$</td>
<td>$q_2$</td>
<td>$q_1$</td>
</tr>
<tr>
<td>$q_1$</td>
<td>$q_1$</td>
<td>$q_2$</td>
</tr>
<tr>
<td>$q_2$</td>
<td>$q_0$</td>
<td>$q_1$</td>
</tr>
</tbody>
</table>

Consider the set $M=\{\delta(q_0,x_0), \delta(q_1,x_0), \delta(q_2,x_0), \delta(q_0,x_1), \delta(q_1,x_1), \delta(q_2,x_1)\}$

**Theorem 2.1**

A non-empty subset $H_M$ of a group $M$ is a subgroup of $G$ iff (i) It is closed under the binary operation in $G$. (ii) The identity $\delta(s_0,x_0) \in H_M \rightarrow \delta(s_0^{-1},x_0^{-1}) \in H_M$. (iii) $\delta(s_0,x_0) \in H_M \rightarrow \delta(s_0^{-1},x_0^{-1}) \in H_M$.

**Proof**

Let $H_M$ be a FSM subgroup of $M$, then by definition it follows that (i) and (ii) hold.

Conversely, Let the given conditions hold in $H_M$. We show $H_M$ forms a FSM Group.

Closure holds in $H_M$ by (i). Again, $\delta(q_a,x_0), \delta(q_b,x_0), \delta(q_0,x_0) \in H_M$.

Thus $H_M$ has identity. Also, for any $\delta(s_0,x_0) \in H_M \rightarrow \delta(s_0^{-1},x_0^{-1}) \in H_M$. (By (1))

Hence associativity holds in $H_M$.

Finally, for any $\gamma(\delta(q_0,x_0), \delta(q_1,x_1)) \in H_M$.

**Theorem 2.2**

A non-empty subset $H_M$ of a FSM Group M is a sub FSM Group of M iff $\delta(q_a,x_0), \delta(q_b,x_0) \in H_M \rightarrow \delta(q_a q_b^{-1}, x_0) \in H_M$.

**Proof**

If $H_M$ is a sub FSM Group of $M$ then, from Figure 1.1 $\delta(s_0,x_0), \delta(s_1,x_1) \in H_M \rightarrow \gamma(\delta(s_0,x_0), \delta(s_1,x_1)) \in H_M$ (follows easily by using definition)

Conversely,

Let the given condition hold in $H_M$. That associativity holds in $H_M$ follows as in previous theorem. Let $\delta(s_0,x_0) \in H_M$ be any element (since $H_M$ is non-empty) Then $\gamma(\delta(s_0,x_0), \delta(s_1,x_1)) \in H_M$.

Thus $H_M$ has identity. Again, for any $\gamma(\delta(s_0,x_0), \delta(s_1,x_1)) \in H_M$.

$\delta(q_a,x_0), \delta(q_b,x_0) \in H_M \rightarrow \delta(q_a q_b^{-1}, x_0) \in H_M$.

$\delta(q_0,x_0) \in H_M \rightarrow \gamma(\delta(s_0,x_0), \delta(s_1,x_1)) \in H_M$.

$\gamma(\delta(s_0,x_0), \delta(s_1,x_1)) \in H_M$.

$\delta(q_0,x_0) \in H_M \rightarrow \gamma(\delta(s_0,x_0), \delta(s_1,x_1)) \in H_M$.

$\gamma(\delta(s_0,x_0), \delta(s_1,x_1)) \in H_M$.

$\gamma(\delta(s_0,x_0), \delta(s_1,x_1)) \in H_M$.

$\gamma(\delta(s_0,x_0), \delta(s_1,x_1)) \in H_M$.

Thus $H_M$ is a sub FSM Group. Hence $H_M$ forms a FSM Group and therefore $H_M$ is a sub FSM Group.

**Theorem 2.3**

If M is a FSM Group and $H_M$ is sub FSM Group of M then $O(H_M)$ divides $O(M)$.

**Proof**

Let $O(M)=n$. Since corresponding to each element in $M$, we can define a right coset of $H_M$ in $M$, the number of distinct right cosets of $H_M$ in $M$ is less than or equal to $n$. Using the properties of equivalence classes, we know $M=\gamma(H_M \delta(s_1,x_1)) \cup \gamma(H_M \delta(s_1,x_1)) \cup \gamma(H_M \delta(s_1,x_1))$.

...
A subgroup H of a group M is called a normal subgroup of M if
\( \delta(q, x_i) H_M = H_M \delta(q, x_i) \) for all \( \delta(q, x_i) \in G_M \).

**Theorem 2.4**

A sub FSM group H_M of a FSM group M is normal subgroup if \( \gamma(\delta(S, x_k)) H_M = H_M \gamma(\delta(S, x_k)) \) for all \( \gamma(\delta(S, x_k)) \in M \).

**Proof**

Let H_M be normal sub FSM group in M.

Then, By definition, \( (H_M \delta(S, x_k)) H_M = H_M \delta(S, x_k) \).

For all \( \delta(S, x_k) \in M \), pre multiply by \( \delta(q, x_i)^{-1} \)
\( \delta(S, x_k)^{-1} H_M \delta(S, x_k) = \delta(S, x_k)^{-1} \delta(S, x_k) H_M = H_M \).

Conversely, \( \gamma(\delta(S, x_k)) H_M \delta(S, x_k) = H_M \delta(S, x_k) \).

Hence H_M is normal.

**Theorem 2.5**

A subgroup H_M of a group M is normal in M iff \( \delta(S, x_k) H_M = H_M \delta(S, x_k) \) for all \( \delta(S, x_k) \in M \).

**Proof**

Let N be a normal in M. Then \( N \gamma(\delta(S, x_k)) = \delta(S, x_k) \gamma N \forall \delta(S, x_k) \in M \).

Let \( \delta(S, x_k) N, \delta(S, x_k) \in M \) be any elements, then
\( \delta(S, x_k) N = \gamma(\delta(S, x_k)) = \gamma(\delta(S, x_k)) \gamma N \).

For all \( \delta(S, x_k) \in M \).

Taking \( \delta(s, x_k) = \delta(s, x_k)^{-1} \). We note as \( \delta(s, x_k) \in M \),
\( \delta(s, x_k)^{-1} \gamma(\delta(S, x_k)) = \delta(S, x_k) \gamma N \forall \delta(S, x_k) \in M \).

**III. PARTIALLY ORDERED FSM GROUPS.**

**Definition 3.1**

A relation \( \rho \) is defined on M as follows \( \delta(q, x_k) \rho \delta(q, x_k) \) if \( \delta(q, x_k) \in M \).

For all \( \delta(q, x_k) \in M \).

**Definition 3.2**

A FSM Group M is partially ordered if there is a partial order \( \rho \) on M that is compatible with the addition modulo in M and for all \( \delta(q, x_k), \delta(q, x_k) \in M \).

**Definition 3.3**

If M is a FSM group and \( \delta(q, x_k), \delta(q, x_k) \in M \), then \( \delta(q, x_k) \) is said to be J-trivial if \( \delta(q, x_k) \) and \( \delta(q, x_k) \) are normal in M for all \( \delta(q, x_k), \delta(q, x_k) \in M \).

**Definition 3.4**

\( \delta(q, x_k) \) and \( \delta(q, x_k) \) are said to be J-equivalent (written \( \delta(q, x_k) \equiv \delta(q, x_k) \)) if \( \delta(q, x_k) \rho \delta(q, x_k) \) and \( \delta(q, x_k) \rho \delta(q, x_k) \).

**Definition 3.5**

M is said to be J-trivial if there are two elements of M which are J-equivalent.

**Theorem 3.1**

A FSM group is J-trivial if and only if it is a quotient of a partially ordered FSM Group.

**Proof**

Let M be a FSM Group write addition Modulo. Assume that M is J-trivial.

I.e. Two elements of M which are J-equivalent are equal.

To prove that, it is a quotient of a partially ordered FSM Group.
(ie.) To prove that, it is a quotient of a FSM Group and it is a partially ordered.

(ie.) to prove M/K is a FSM Group and M/K is partially ordered.

Case1

If K is a sub FSM Group, then it must be a normal FSM Group. [Every sub FSM Group of an abelian FSM Group is normal FSM group]. It’s clear that M/K is quotient FSM Group.

Case2

If K is not a sub FSM Group, from the definition of centre of FSM Group, there exists atleast one element belongs to Z(M), then the corresponding normalizer is the Sub FSM Group. Definitely it will be the normal FSM Group. From the above two cases, clearly M/K is a quotient FSM Group.

It is enough to prove M/K is partially ordered.

Elements of M/K of is of the form (K δ(qi,xk) \ δ(qa,xk) \in M) which is the FSM Group. From the definition of the partially ordered implies M is the partially ordered set.

Therefore, M is a quotient of a partially ordered FSM Group.

**Definition 3.6 (Permutation)**

Let M be a non empty set. Any 1-1,onto mapping f: M \rightarrow M is called permutation (or a non –singular transformation) of M.

We shall use the notation A(G_M) to denote the set of all permutation.

**Example 3.1**

Consider the machine M whose transition table is given below.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( a )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_0 )</td>
<td>( s_0 )</td>
<td>( s_1 )</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>( s_1 )</td>
<td>( s_2 )</td>
</tr>
</tbody>
</table>

Let \( G_M = \{ \delta(s_0,a), \delta(s_1,a), \delta(s_2,a), \delta(s_0,b), \delta(s_1,b) \} \), \( \delta(s_2,b) \)

Consider the map \( f: G_M \rightarrow G_M \) such that

\( f(\delta(s_0,a)) = \delta(s_2,b) \), \( f(\delta(s_1,a)) = \delta(s_0,b) \), \( f(\delta(s_2,a)) = \delta(s_1,a) \), \( f(\delta(s_0,b)) = \delta(s_1,b) \)

\( g: G_M \rightarrow G_M \) such that

\( g(\delta(s_0,a)) = \delta(s_2,a) \), \( g(\delta(s_1,a)) = \delta(s_0,a) \), \( g(\delta(s_2,a)) = \delta(s_1,a) \), \( g(\delta(s_0,b)) = \delta(s_1,b) \)

then both \( f, g \) are permutations on \( G_M \). Again the identity map \( I: G_M \rightarrow G_M \) such that \( I(\delta(s_i,a)) = (s_i,i) \), for all \( \delta(s_i,i) \in G_M \) is also a permutation. It is easy to check that maps \( f \circ g, g \circ f \) and \( f \circ f \) are also permutation of \( G_M \).

**Definition 3.7 (Kernel of f)**

Let \( f: M_1 \rightarrow M_2 \) be a homomorphism. The kernel of \( f \), (denoted by Ker f) is defined by

\[ \text{Ker } f = \{ \delta(s_i,a_k) \in G_M / f(\delta(s_i,a_k)) = \delta'(s_e, a_e) \} \]

**Definition 3.8 (homomorphism)**

Let \( M_1, \gamma \) and \( M_2, \gamma \) be two groups. A mapping \( f: M_1 \rightarrow M_2 \) is called a homomorphism if

\[ f(\delta(q_i,a_k) \gamma \delta(q_i,a_l)) = f(\delta(q_i,a_k)) \gamma f(\delta(q_i,a_l)) \]

where \( \delta(q_i,a_k), \delta(q_i,a_l) \in G_M \).

\( f \), in addition \( f \) happens to be one-one, onto, we say \( f \) is an isomorphism and it is denoted by \( M \cong M \). An onto homomorphism is called epimorphism. A one-one homomorphism is called monomorphism. A homomorphism form a group \( G_M \) to itself is called an endomorphism of \( G_M \). An isomorphism form a group \( G_M \) to itself is called an automorphism of \( G_M \). If \( f: G_M \rightarrow G_M \) is onto homomorphism then \( G_M \) is called homomorphic image of \( G_M \).

Clearly \( M \) and \( \{ \delta(s_i,a_k) \} \) are normal sub FSM Groups of \( M \) and it is called as the trivial normal sub FSM Groups.

A FSM Group not having any non-trivial normal sub FSM Groups is called a simple FSM Group. It is easy to see that if \( H_M \) is a normal sub FSM Group of \( M \) and \( N \) is a sub FSM Group of \( M \) such that \( H_M \subseteq N \subseteq M \) then \( H_M \) in \( N \).

\[ \Rightarrow \delta(S_i, x_k)[\delta(S_j^{-1}, x_k^{-1})] \quad H_M \delta(S_i, x_k) = \delta(S_i, x_k) H_M \]

\[ [\delta(S_i, x_k)\delta(S_j^{-1}, x_k^{-1})] \quad H_M \delta(S_i, x_k) = \delta(S_i, x_k) H_M \]

\[ H_M \delta(S_i, x_k) \quad \delta(S_i, x_k) H_M \]

Hence \( H_M \) is normal

**Model 3.1**

A three-state finite state machine \( M_1 = \{ Q, \Sigma, \Delta, q_0, \lambda \} \) with \( q = \{ q_0, q_1, q_2 \} \),

\( \Sigma = \{ x_0, x_1 \}, \Delta = \{ y_0, y_1 \} \) and the next state function \( \delta_1 \) is defined by TABLE V and \( M_2 = \{ Q, \Sigma, \Delta, q_0, \lambda \} \) with \( Q = \{ q_0, q_1, q_2 \}, \Sigma = \{ x_0, x_1 \}, \Delta = \{ y_0, y_1 \} \) and the next state function \( \delta_2 \) is defined by TABLE VI. From Figure 1.1 ,

**TABLE IV**

<table>
<thead>
<tr>
<th>( N )</th>
<th>( a )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_0 )</td>
<td>( s_0 )</td>
<td>( s_1 )</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>( s_1 )</td>
<td>( s_2 )</td>
</tr>
</tbody>
</table>

**TABLE V**

<table>
<thead>
<tr>
<th>State</th>
<th>( \delta_1 )</th>
<th>( \delta_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input ( x_0 )</td>
<td>Input ( x_1 )</td>
<td>Input ( x_0 )</td>
</tr>
<tr>
<td>( q_0 )</td>
<td>( q_2 )</td>
<td>( q_0 )</td>
</tr>
<tr>
<td>( q_1 )</td>
<td>( q_0 )</td>
<td>( q_2 )</td>
</tr>
<tr>
<td>( q_2 )</td>
<td>( q_0 )</td>
<td>( q_2 )</td>
</tr>
</tbody>
</table>

We define function \( \Phi \) from \( M_1 \) to \( M_2 \) by

\[ \Phi(\delta_1(q_i,x_j)) = \delta_2(q_i,x_j) \gamma \delta_2(q_i,x_j) \text{ for all } \delta_1(q_i,x_j) \in M_1 \text{ and } \delta_2(q_i,x_j) \in M_2. \]

Let \( M_1 = \{ \delta_1(q_0,x_0), \delta_1(q_1,x_0), \delta_1(q_2,x_0), \delta_1(q_0,x_1), \delta_1(q_1,x_1) \delta_1(q_2,x_1) \} \) and
\(M_2 = \{ \delta_2(q_0,x_0), \delta_2(q_1,x_0), \delta_2(q_2,x_0), \delta_2(q_0,x_1), \delta_2(q_1,x_1), \delta_2(q_1,x_1) \} \).

To check \( \phi \) is a homomorphism

\( (ie) \phi(\delta(q_i,x_i)\gamma(\delta(q_k,x_k))) = \phi(\delta(q_i,x_i))\gamma(\phi(\delta(q_k,x_k))) \)

consider, \( \delta_1(q_0,x_0), \delta_1(q_1,x_1) \in M_1 \) \( \delta_2(q_2,x_1) \in M_2 \).

To prove: \( \phi(\delta_1(q_0,x_0)\gamma(\delta_2(q_2,x_1))) = \delta_1(q_0,x_0)\gamma(\delta_2(q_2,x_1)) \)

\( \phi(\delta_1(q_0,x_0))\gamma(\phi(\delta_2(q_2,x_1))) = \delta_1(q_0,x_0)\gamma(\delta_2(q_2,x_1)) \)

\( \phi(\delta_1(q_0,x_0)) = \delta_1(q_0,x_0) \)

\( \phi(\delta_2(q_2,x_1)) = \delta_2(q_2,x_1) \)

\( \phi(\delta_1(q_0,x_0)\gamma(\delta_2(q_2,x_1))) = \delta_1(q_0,x_0)\gamma(\delta_2(q_2,x_1)) \)

\( \phi(\delta_1(q_0,x_0))\gamma(\delta_2(q_2,x_1)) = \delta_1(q_0,x_0)\gamma(\delta_2(q_2,x_1)) \)

From (5) and (6) we have,

\( \phi(\delta_1(q_0,x_0))\gamma(\delta_2(q_2,x_1)) = \delta_1(q_0,x_0)\gamma(\delta_2(q_2,x_1)) \).

Hence \( \phi \) is a homomorphism from \( M_1 \) to \( M_2 \).

**Definition 3.9 Ordered homomorphism:**

A homomorphism of ordered groups \( \phi: M \rightarrow N \) is a group homomorphism from \( M \) into \( N \) such that , for every \( \delta(q_i,x_i) \in M \), \( \rho \delta(q_i,x_i) \rho \delta(q_i,x_i) \) \( I. \)

**IV. ORDERED IDEALS FSM GROUP**

**Definition 4.1:**

An order ideal of an ordered FSM group \( S \) is a subset \( I \) of \( S \) such that if \( \delta(q,x), \rho \delta(q,x) \) and \( \delta(q,x) \in I \), \( \delta(q,x) \in I \) is an order ideal generated by \( I. \)

**Order ideal generated by an element**

The order ideal generated by an element \( \delta(q,i) \)

Is the set of all \( \delta(q_i,x_i) \rho \delta(q_i,x_i) \) and it is denoted by \( \downarrow \delta(q,i) \)

\( (ie) \downarrow \delta(q,i) = \{ \delta(q_i,x_i) \in S / \delta(q_i,x_i) \rho \delta(q_i,x_i) \} \)

**Result 4.1:**

1. The intersection of any family of order ideals is also an order ideal.
2. The union of any family of order ideals is also an order ideal.
3. If \( I \) is an order ideal and \( K \) is an arbitrary subset of \( S \), then the left quotient \( K^{-1}I \) and the right quotient \( IK^{-1} \) are also ideals.

**V. RECOGNITION BY ORDERED FSM GROUP**

**Definition 5.1**

Let \( \phi: S \rightarrow T \) be a surjective homomorphism of ordered FSM groups. A subset \( Q \) of \( S \) is recognized by \( \phi \) if there exists an order ideal \( P \) of \( T \) such that \( Q = \phi^{-1}(p) \). This condition implies that \( Q \) is an order ideal of \( S \) and that \( \phi(Q) = \phi(p) \).

**Definition 5.2**

A subset \( Q \) of \( S \) is said to be recognized by an ordered FSM group \( T \) if there exists a surjective homomorphism of ordered FSM groups from \( S \) onto \( T \) that recognizes \( Q \).

**Proposition 5.1**