Abstract—In this paper, we examine the heat distribution that is propagated within a solar panel with a radiometric heat source that approximates incoming solar irradiance where simultaneous convective and radiative heat transfer losses are present in the interior. The solar panel receiver is modeled as a two dimensional partial differential equation with nonlinear heat source/sink terms with a specified Dirichlet boundary condition which models the presence of a heat exchanger with molten liquid salt as a circulating fluid. We investigate the resultant system of nonlinear equations which are developed in terms of a radial basis function (RBF) discretization approach. Operating parameters for the thermal energy recovery with respect to local atmospheric conditions such as wind speed and solar irradiation are investigated in terms of optimal heat transfer. Further applications of the developed analysis technique for material thermal conductivity testing are discussed.

Keywords—Concentrated solar power, inverse heat conduction problem, Radial Basis Function (RBF), material thermal conductivity testing.

I. INTRODUCTION

Solar energy plants commonly employ concentrated solar power (CSP) designs for thermal energy recovery which entails the use of an array of ground based mirrors which reflect/focus incoming solar irradiation onto a central elevated tower from which the received energy may be captured through heat transfer into a circulating fluid. The captured heat may then be either immediately used to generate steam to power turbines for electrical power generation, or alternately simply stored in thermal storage tanks such as molten salt baths for later utilization [1].

Generally the analysis in such solar energy systems for the receiving solar panels is considered as an uncoupled radiative heat transfer problem, however in practice heat losses occur mainly from a mixture of convective and radiative heat transfer back into the surrounding atmosphere. Whilst the design of CSP systems as a field of research continues to evolve with regards to panel geometry and configurations for the purposes of analysis in this paper we will develop a general mathematical formulation which may be applied to specific geometries/configurations. The general form of a solar receiving/reflecting panel with convective and radiative heat transfer mechanisms is illustrated in figure 1.

For the purposes of analysis we will consider the physical case of a compound parabolic as discussed in [2] where a circulating molten salt mixture is in contact with the solar panel thus providing a constant temperature at the boundary of the panel.

The operating principle of the problem as illustrated in figure 2 is that the panel which is insulated from below...
receives the incident solar heat flux from above, then the heat is propagated through the area with some heat dissipation through combined convective/radiative heat losses to the boundary, from where the remaining heat is transported from a circulating fluid heat exchanger.

II. MATHEMATICAL MODEL OF EXPERIMENTAL SYSTEM

We consider the physical system under consideration which has a mathematical representation as a flat two dimensional plate for which a possibly inhomogeneous thermal conductivity $k(x,y)$ is known. A domain $\Omega \subset \mathbb{R}^2$ with a boundary $\Gamma = \partial \Omega$ consists of the plate under test to which a heat source $q(x,y)$ is applied and the domain is subjected to a specified Dirichlet boundary condition in terms of a given fixed temperature $T_f(x,y)$.

From the literature study cited in [3] typical operating values for molten salt in CSP designs will lie in the range 290 ºC to avoid salt freezing to 565 ºC to avoid solar salt decomposition. Above an upper limit of approximately 580 ºC thermal decomposition will occur and high corrosive chemicals such as nitrites and peroxides start to form. The thermal decomposition will occur and high corrosive chemicals such as nitrites and peroxides start to form. The starting point in the analysis is the energy equation

$$0 = \nabla \cdot (k \nabla T) - h(T - T_a) - \varepsilon \sigma (T^4 - T^4) + q, \quad \forall \mathbf{r} \in \Omega \setminus \Gamma$$
$$T_f = T, \forall \mathbf{r} \in \Gamma$$

From energy conservation principles heat dissipation necessary to avoid overheating of the sample is modeled in terms of coupled convective and radiative terms. The convection term uses Newton’s law of cooling with a known convective heat transfer coefficient $h(x,y)$ and the radiative loss is assumed to follow a simple Stefan–Boltzmann law with an effective plate total emissivity $\varepsilon(x,y)$ where the far field temperature above the plate is $T_{\infty}$ which is the so called “sky temperature” expressed as

$$T_{\infty} = 0.05527T_a^{0.5}$$

where $T_a$ is the ambient temperature [4]. In general values for the convective heat transfer coefficient $h$ are difficult to determine since $h$ is due to a combination of free convection, forced convection and additional factors such as wind speed and direction.

Based on investigations reported in [5] the natural convection $h_{nc}$ and forced convection $h_{fc}$ for horizontal plates at a surface temperature $T$ may be estimated as

$$h_{nc} = a \left[ \frac{\mu T_m}{\rho (T - T_a) \sigma k^2 T^2} \right]^{-1/3}$$
$$h_{fc} = b + c v_w$$

where $a = 0.155$ corresponds to the case for an upwards horizontal surface that is maintained at a constant surface temperature for simplicity, $b = 8.55$ and $c = 2.56$ are experimental parameters where $v_w$ is the prevailing wind velocity, and the thermo-physical properties for the air viscosity $\mu$, specific heat capacity $c_p$, thermal conductivity $k$ and mass density $\rho$ are all evaluated at the mean air temperature defined as $T_m = \frac{1}{2} (T + T_{\infty})$.

Expressions to determine the thermo-physical properties are necessary when calculating the natural convection heat transfer. For dry air the density is calculated as $\rho = p/(RT)$ where $p$ is the air pressure, $T$ is the thermodynamic temperature of the corresponding air medium, and $R = 287.058 \text{ J.kg}^{-1}.\text{K}^{-1}$ is the specific gas constant for dry air.

To a reasonable degree of accuracy of approximately ±2% we may apply a Sutherland formula for the viscosity and thermal conductivity of a fluid as

$$\mu = \mu_0 \left( \frac{T}{T_0 \mu} \right)^{3/2} \frac{T_0 + S_\mu}{T + S_\mu}$$
$$k = k_0 \left( \frac{T}{T_0 k} \right)^{3/2} \frac{T_0 + S_k}{T + S_k}$$

where for dry air $T_0 = 273 \text{ K}$, $\mu_0 = 1.716 \times 10^{-5} \text{ N.s.m}^{-2}$, $S_\mu = 111 \text{ K}$, $T_0 k = 273 \text{ K}$, $k_0 = 0.0241 \text{ W.m}^{-1}.\text{K}^{-1}$, and $S_k = 194 \text{ K}$ following [6].

The specific heat capacity for air is generally reported in tabular form and due to the potentially large values of incident heat flux in solar power systems from focused arrays of mirrors in CSP designs a large range of air temperatures is possible, ranging from a temperature of order of magnitude of 1000 ºC at the interface between the metal receiver plates and the air in contact to ambient temperature of order of magnitude of 20 ºC. As a result to specify $c_p$ we opt for a low order polynomial curve fit performed in MATLAB of reported experimental data in [7] for the temperature range from below ambient of 250 K to 1700 K, as

$$c_p = \alpha + \beta T + \gamma T^2$$

with $\alpha = 9.42359 \times 10^2$, $\beta = 0.193172$, and $\gamma = -9.042405 \times 10^{-7}$. The total convection heat transfer coefficient $h$ i.e. the combination of the natural convection and the forced convection is calculated as

$$(Nu)^n = (Nu)^n_{nc} + (Nu)^n_{nc}$$
term $\dot{q}$ in the energy equation that is introduced from the received solar irradiance is necessary and this information has been reported in [2] for a CSP design which reports heat fluxes in the range from zero to 1.6 MW.m$^{-2}$ for distances from zero to 13 m from the focal point on the receiver.

In order to quantify this heat flux, which is the resultant of a combination of an array of receivers/mirrors that focuses the total received solar irradiance, we use a simple total radiation equation $Q = \sigma T^4$ where $Q$ is the radiant power in units of [W.m$^{-2}$] and $\sigma = 5.67 \times 10^{-8}$ W.m$^{-2}$.K$^{-4}$ is the Stefan–Boltzmann constant. Applying this equation yields an equivalent thermodynamic temperature of $T = 1938.1$ K for a heat flux of 0.8 MW.m$^{-2}$. Based on the heat flux versus distance relationship for the reported CSP system we will model the heat source term $\dot{q}$ as the heat flux introduced by a blackbody source.

This approach is consistent with existing laboratory based experimental heat flux techniques [8] and to avoid complicating radiometry size-of-source effects (SSE) for radiation calculations as discussed in [9] we will assume that the heat flux is introduced by a cylindrical blackbody 25 mm in diameter and 100 mm in length and develop the beam propagation from the blackbody source to the target in terms of the irradiance.

To fully quantify the irradiance from the blackbody we assume a graphite cavity with a material emissivity of $\varepsilon_m = 0.9$ so that an apparent blackbody cavity of $\varepsilon_{BB} = 0.998$ is present [10], and assume a two dimensional Gaussian spatial distribution of the emitted irradiance as

$$\dot{q} = \alpha_s \varepsilon_{BB} \sigma T^4_{BB} \exp \left[ -\frac{(x-x_{ob})^2}{\sigma_x^2} - \frac{(y-y_{ob})^2}{\sigma_y^2} \right]$$

(10)

where $x_{ob}, y_{ob}$ are the spatial coordinates where the irradiance is focused on the receiver, $\sigma_x$ and $\sigma_y$ are parameters that specify the spread on the distributed irradiance, and $\alpha_s$ is an absorptivity term of the domain.

For the purposes of an initial simulation we will consider an aluminum receiver plate with material properties specified in table 1 which has a higher thermal conductivity to more clearly observe the heat flow in the domain, although in practice the use of stainless steel alloys is more common. In actual solar designs the reflecting panels should have a high reflectivity however the consequence of this is a lower emissivity.

### III. Numerical Formulation

Let $\kappa(x,y)$ be the known thermal conductivity, and $\theta(x,y)$ the computational solution of the temperature of the receiver surface respectively. Since we wish to develop a general mathematical and numerical implementation that may be applied to various geometries and configurations we utilize a radial basis function (RBF) approach as discussed in [13].

In our implementation the unknown function $\theta(x,y)$ is constructed as a linear combination of underlying basis functions, where following [14] we opt for the use of a Gaussian RBF with a shape parameter $s$ so that with $r = \sqrt{(x-x_j)^2 + (y-y_j)^2}$ we have

$$\theta(x, y) = \sum_{j=1}^{N} \lambda_j \phi(x, y)$$

(11)

$$\phi(x, y) = \exp \left(-s^2 \left[ (x-x_j)^2 + (y-y_j)^2 \right] \right)$$

(12)

Noting that the spatial derivatives of a RBF may be constructed by application of the calculus chain rule as $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial x}$ and $\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial y}$, it follows that

$$\frac{\partial \theta}{\partial x} = \sum_{j=1}^{N} \lambda_j \phi(x, y)$$

(13)

$$\frac{\partial \theta}{\partial y} = \sum_{j=1}^{N} \lambda_j \phi(x, y)$$

(14)

$$\frac{\partial^2 \theta}{\partial x^2} = \sum_{j=1}^{N} \left[ \frac{2s^2}{\sqrt{\pi s^2 \sigma_x^2}} \left( (x-x_j)^2 \right) \right] \lambda_j \phi(x, y)$$

(16)

$$\frac{\partial^2 \theta}{\partial y^2} = \sum_{j=1}^{N} \left[ \frac{2s^2}{\sqrt{\pi s^2 \sigma_y^2}} \left( (y-y_j)^2 \right) \right] \lambda_j \phi(x, y)$$

(17)

To perform a simulation choose $N_p$ points within the two dimensional domain $\Omega$, using for example a sampling scheme such as a Halton distribution or a simple random distribution of 2D points, and $N_B$ points on the boundary $\Gamma$ so that the total number of points is $N = N_p + N_B$. Substitute the discretization into the energy equation for the $N_p$ points in the interior, and then apply the boundary condition $\theta(x, y) = T_f(x, y) \forall r = (x, y) \in \Gamma$ for the $N_B$ points on the boundary which will yield a set of simultaneous nonlinear equations $\Phi(A) = 0$ where $A = (\lambda_1, ..., \lambda_N)$ is a vector argument that must be determined.

For simplicity construct the functions $A = \theta, B = \partial \theta / \partial x, C = \partial \theta / \partial y, D = \partial^2 \theta / \partial x^2, E = \partial^2 \theta / \partial y^2, F = \varepsilon \sigma T^4_{BB} + \dot{q}$ which will all have the unknown coefficients $\lambda_1, ..., \lambda_N$ of the RBF’s $\phi(r)$ as parameters and the spatial locations $r = (x, y)$ as arguments so that

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Quantity</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{mp}$</td>
<td>Melting point</td>
<td>775 K</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Mass density</td>
<td>2770 kg.m$^{-3}$</td>
</tr>
<tr>
<td>$c_p$</td>
<td>Specific heat capacity</td>
<td>875 J.kg$^{-1}$.K$^{-1}$</td>
</tr>
<tr>
<td>$k$</td>
<td>Thermal conductivity</td>
<td>177 W.m$^{-1}$.K$^{-1}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Thermal diffusivity</td>
<td>$73 \times 10^{-8}$ m$^2$.s$^{-1}$</td>
</tr>
</tbody>
</table>
\[ \Phi_i = \frac{\partial \kappa}{\partial x} B + \frac{\partial \kappa}{\partial y} C + \kappa(D + E) - h(A - T_0) - \varepsilon \sigma A^4 + F, i = 1, ..., N_i \] (18)

\[ \Phi_i = A - T_f, i = N_i + 1, ..., N_i + N_B \] (19)

This set of equations may be solved using a standard Brodyen technique for systems of nonlinear equations with the aid of a Scherma-Morrison formula for the Brodyen approximation to the Jacobian as discussed in [15]. In order to generate a starting solution we solve the equivalent system of \( \Phi(A) = 0 \) for pure conduction in the absence of convective/radiative heat losses with the same boundary conditions and which may be written as a matrix equation

\[ M \lambda_0 = Y \] (20)

Due to the algebraic complexity of the underlying system and space limitations we will omit the inclusion of the details. The main numerical challenge in applying the RBF approach is in the correct determination of the shape parameter \( s \). At the present time there is no general consensus within the mathematical literature as to the correct specification of the shape parameter and the situation is in a certain sense analogous to the correct choice of type of and quality of meshes in finite element method (FEM) simulations.

Since radiative heat transfer problems are nonlinear in nature there is no mathematical mechanism to formally determine the uniqueness of numerical solutions of the underlying PDE’s. The situation in RBF studies for the existence problem, commonly discussed within the research literature in terms of the invertibility of the associated Kansa matrix for RBF collocation schemes, is slightly different.

In meshless computing a function \( u(x) \), \( x \in \mathbb{R}^n, n \in \mathbb{N} \), which is to be either fitted to known values or which is used as an approximation in a PDE discretization scheme, is constructed as a linear combination of underlying basis functions of form \( u(x) = \sum_{j=1}^{n} \lambda_j \phi(x) \).

For the particular case of RBF schemes different choices are possible such as the Gaussian which we have already considered, but in addition other popular choices such as the inverse multiquadric, regular multiquadric and Matern functions are also possible where the essential criteria is that the basis function is positive definite as discussed in [14].

<table>
<thead>
<tr>
<th>Types of Different Choices of Radial Basis Functions with a Shape Parameter ( s ) (Adapted from Reference [16])</th>
<th>Table II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Piecewise smooth</td>
<td>( \phi(r) )</td>
</tr>
<tr>
<td>Piecewise polynomial</td>
<td>( r^n, n ) odd</td>
</tr>
<tr>
<td>Thin plate spline</td>
<td>( r^{n \log(c)} n ) even</td>
</tr>
<tr>
<td>Multiquadric</td>
<td>( \sqrt{1 + (cr)^2} )</td>
</tr>
<tr>
<td>Inverse multiquadric</td>
<td>( 1/\sqrt{1 + (cr)^2} )</td>
</tr>
<tr>
<td>Inverse quadratic</td>
<td>( 1/\sqrt{1 + (cr)^2} )</td>
</tr>
<tr>
<td>Gaussian</td>
<td>( e^{-c(r)^2} )</td>
</tr>
</tbody>
</table>

The algorithm implementation may be specified as follows:

1. Load the boundary data \( F \) and extract vectors for the \( x \)-data as \( x_b \), for the \( y \)-data as \( y_b \), for the boundary temperature as \( T_f \) and the number of boundary data points \( N_B \)
2. Load the interior data \( \Omega \) and extract vectors for the \( x \)-data as \( x_i \), for the \( y \)-data as \( y_i \), and the number of interior points \( N_i \)
3. Work out the total number of points \( N = N_i + N_B \)
4. Solve the heat conduction equation \( \nabla \cdot (\kappa \nabla \theta) + \dot{q} = 0 \forall r \in \Omega \) such that \( \theta = T_f \forall r \in \Gamma \) in terms of the matrix equation \( M \lambda_0 = Y \) to obtain \( \lambda_0 \)
5. Step through the interior points \( r \in \Omega \) to build up \( \Phi_i = (\partial_x \kappa)B + (\partial_y \kappa)C + \kappa(D + E) - h(A - T_0) - \varepsilon \sigma A^4 + F \) for \( i = 1, ..., N_i \) i.e. for each of the \( N_i \) points
6. Step through the boundary points \( r \in \Gamma \) to build up \( \Phi_i = A - T_f \) for \( i = 1, ..., N_B \) i.e. for each of the \( N_B \) points
7. Collect the associated functions \( \Phi_i (i = 1, ..., N_i) \) and \( \Phi_i (i = 1, ..., N_B) \) and form \( \Phi(A) = 0 \) and solve the \( N \) simultaneous equations with a starting solution \( \lambda_0 \) for the final solution \( \lambda = (\lambda_1, ..., \lambda_N) \)

In the above algorithm when calculating the convective heat transfer coefficient \( h \) we utilize the particular choice of \( \lambda = (\lambda_1, ..., \lambda_N) \) to logically carry through in the thermal calculations.

An example when calculating the mean temperature we specify \( T_m = \frac{1}{2} (\Sigma_{j=1}^{N} \lambda_1 \phi + T_0) \), and similarly in computing the corresponding thermo-physical parameters such as the specific heat we specify \( c_p = \alpha + \beta (\Sigma_{j=1}^{N} \lambda_1 \phi)^2 \).

The benefit of this approach is that no underlying assumptions as to the homogeneity of the underlying fields such as the thermal conductivity are necessary and the only data requirements are for the corresponding nodal locations.

IV. DISCUSSION

An example for the methodology developed in this paper for solving the heat distribution in a solar panel with convective/radiative heat losses is illustrated in figure 3 which is for a flat polygon which lies in the \( xy \)-plane with \( N_B = 4 \) boundary nodes and \( N_i = 12 \) irregular spaced interior nodes, so that the total number of unknown coefficients is \( N = 16 \).

For the temperature specifications we assume a fluid temperature of \( T_f = 575 \text{ K} \), an ambient temperature of \( T_0 = 300 \text{ K} \), and for simplicity a homogenous thermal conductivity of \( \kappa = 177 \text{ W} \text{ m}^{-2} \text{ K}^{-1} \).

For a full simulation the nonlinear system of coupled equations \( \Phi_i \) must be simultaneously solved for using an approximate value of \( \lambda_0 \) which is a solution of the associated generalized Poisson heat conduction equation, with appropriate heat source/sink terms.

Due to numerical difficulties in solving systems of simultaneous nonlinear equations in our particular problem we make the observation that a spatial visualization of the system
error using the initial conductivity of the sample. The inverse heat conduction problem to determine the thermal methodology presented in this paper may also be applied as an using inverse theory methods as discussed in [18]. We make the observation that if the temperature distribution is experimentally measured from a thermal imager then the formulae for counter-flow heat exchangers as discussed in [17].

An example of this may be seen in figure 3 which plots the residual within the domain $\Omega$ at each of the $N_f = 12$ interior points.

Fig. 3 Illustration of RBF implementation for combined heat transfer problem for a wind velocity of $v_w = 5.88$ m/s, a constant boundary temperature of $T_f = 540$ K at the $N_B = 4$ boundary points, an asymmetric heat source term, with a visualization of the system frames, which is the corresponding equation for the point $r_i = (x_i, y_i) \in \Omega$ for $i \in [1, \ldots, N_f]$, may be utilized as a convenient tool to check the accuracy.

An example of this may be seen in figure 3 which plots the error using the initial $A_0$ value; where for the correct choice of $A$ the surface $\Phi$ will coincide with the $xy$-plane.

Parametric studies for the optimal heat transfer for operating conditions/specifications may be investigated using the formulae for counter-flow heat exchangers as discussed in [17].

We make the observation that if the temperature distribution is experimentally measured from a thermal imager then utilizing inverse theory methods as discussed in [18] that the methodology presented in this paper may also be applied as an inverse heat conduction problem to determine the thermal conductivity of the sample.

REFERENCES