Thermoelastic Buckling Analysis of Power-law, Sigmoid, Exponential FGM Circular Plates with Piezoelectric Actuators

A. R. Khorshidvand, J. Sajedi, and M. Javadi

Abstract—In this paper, buckling of elastic, circular plates made of functionally graded material with surface-bounded piezoelectric layers subjected to thermal loading have been investigated. Boundary condition of the plate as immovable clamped edge is considered. The material properties of the FG plates except poisson’s ratios are assumed to vary continuously throughout the thickness direction according to the volume fraction of constituents defined by power-law, sigmoid, and exponential function. The nonlinear equilibrium equations are derived based on the classical plate theory using variational formulations and then linear stability equations are used to obtain the critical buckling of solid FG circular plate under thermal load as uniform temperature rise. The effects of piezoelectric actuators on buckling of plate P-, S-, E-FGM are compared. The results are validated with the known data in the literature.

Keywords—Classical plate theory, Functionally graded material, Thermal buckling.

I. INTRODUCTION

Many studies are reported on buckling and bending behavior of FGM structures. Axisymmetric bending of functionally graded circular and annular plates is studied by Reddy et al. [1]. They presented the solutions for deflections and force and moment resultants based on the first-order plate theory in terms of those obtained using the classical plate theory. The buckling analysis of circular orthotropic plates under thermal loads is given by Najafizadeh and Eslami [2]. Khorshidvand et al. [3] presented buckling analysis of circular FGM plate integrated with piezoelectric layers subjected to three kinds of thermal loadings based on classical plate theory. Lanhe [7] obtained the closed form solution for the thermal buckling of functionally graded rectangular simply supported plates subjected to two types of temperature fields; uniform temperature rise and gradient across the thickness of the plate, employing the first-order shear deformation theory.

The present paper deals with determination of the stability problem and presents closed-form solutions for critical buckling temperature of Piezoelectric P-, S-, E-FGM circular plate, which are subjected to uniform temperature rise. Clamped edge boundary condition is assumed for the plate.

II. DERIVATION OF GOVERNING EQUATIONS

Consider a uniform thin circular plate made of FGM, as shown in Figure (1). To extract formulations, a cylindrical coordinate system is taken in the center of plate’s middle plane. The FGM profile across the thickness direction of the plate, made of ceramic and metal constituent materials, may be assumed to follow a function form as P-FGM plates as

\[
Pr(z) = \frac{Pr_{cm} + Pr_{cm} (2z + h) - 2zh}{2h}
\]

The value of n, power law index, equal to zero represents a fully ceramic plate. Two power law functions S-FGM plates as

\[
Pr(z) = \frac{Pr_m + Pr_m [1 - \frac{1}{2} \left(\frac{h-2z}{h}\right)^{n}]}{2h} \quad \text{for} \quad 0 \leq z \leq h/2
\]

and exponential function E-FGM plates

\[
Pr(z) = A e^{\frac{B (h+2z)}{h}} \quad \text{for} \quad -h/2 \leq z \leq 0
\]

Where \( Pr_m, Pr_c \) denote any material property of the FGM, metal, and ceramic; such as the modulus of elasticity E and the coefficient of thermal expansion \( \alpha \). The relations (1), (2), and (3) indicate that the top surface of the plate \( (z = h/2) \) is ceramic-rich whereas the bottom surface \( (z = -h/2) \) of the plate is metal-rich. Generally, Poisson’s ratio is assumed constant across the plate thickness.

A. Basic Equations

The material properties are assumed to be independent of temperature, and the stress and strain relations are linear. The constitutive relations of functionally graded materials in thermal environment for the plane-stress condition are written as

\[
\sigma_{\tau} = E(z) (\varepsilon_{\tau} + \nu_{\tau 0}) \frac{(1-\nu)}{(1-\nu^2)} - E(z) \alpha(z) T(z) \frac{(1-\nu)}{(1-\nu^2)}
\]

\[
\sigma_{\theta 0} = E(z) (\varepsilon_{\theta 0} + \nu_{\tau 0}) \frac{(1-\nu)}{(1-\nu^2)} - E(z) \alpha(z) T(z) \frac{(1-\nu)}{(1-\nu^2)}
\]
\[ \sigma_{r0} = E(z)/(1 + \nu) e_{r0} \]  

(6)

The plate is assumed to be comparatively thin, and according to the Love-Kirchhoff assumptions, shear deformations normal to the plate are disregarded. Using the classical plate theory (CPT), strain components at distance \( z \) from the middle plane are given in matrix form as [4]

\[
\begin{bmatrix}
\varepsilon_{r0} \\
\varepsilon_{00} \\
\gamma_{r0}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{2}(u_r + \frac{1}{2}(w_r))^2 \\
\frac{1}{2}(u_0 + v_0 - \frac{1}{2}(w_0))^2 \\
\frac{1}{2}(u_0 + v_0 - \frac{1}{2}(w_0))^2
\end{bmatrix} + z
\begin{bmatrix}
-w_{rr} \\
-w_{0r} + \frac{1}{2}w_{00} \\
-w_{0r} + \frac{1}{2}w_{00}
\end{bmatrix}
\]

where a comma in subscript indicates partial differentiation and where \( \varepsilon_{rr}, \varepsilon_{00}, \gamma_{r0} \) are the strain components along the \( r-, \theta-, \) and \( z- \) directions, respectively. The stress components in plane-stress condition in the plate (superscript \( h \)) are written as following

\[
\begin{bmatrix}
\sigma_{rh} \\
\sigma_{0h} \\
\gamma_{rh}
\end{bmatrix} =
\begin{bmatrix}
Q_11 \\
Q_21 \\
0
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{r0} - \alpha(z)T(z) \\
\varepsilon_{00} - \alpha(z)T(z) \\
\gamma_{r0}
\end{bmatrix}
\]

(8)

Where the plane-stress-reduced stiffness is defined as

\[ Q_{11} = Q_{22} = \frac{E(z)}{1 - \nu^2}, \quad Q_{21} = Q_{11} = \frac{E(z)}{2(1 + \nu)} \]

And stress components in piezoelectric parts of the plate are written as following

\[
\begin{bmatrix}
\sigma_{rh} \\
\sigma_{0h} \\
\gamma_{rh}
\end{bmatrix} =
\begin{bmatrix}
\sigma_{rh} \\
\sigma_{0h} \\
\gamma_{rh}
\end{bmatrix} + \begin{bmatrix}
\sigma_{rh} \\
\sigma_{0h} \\
\gamma_{rh}
\end{bmatrix} - \begin{bmatrix}
\sigma_{rh} \\
\sigma_{0h} \\
\gamma_{rh}
\end{bmatrix}
\]

(9)

The total potential energy for piezoelectric FG circular plate can be written as follows

\[ U = U^p + U^h \]

\[ U^p = \frac{1}{2} \int_0^R \int_0^{2\pi} \left[ (e - \alpha P) (\Delta T) \right] [e - \alpha P] (\Delta T) \cdot [E]^T [k] [E] \cdot r \cdot dr \cdot d\theta \]

\[ U^h = \frac{1}{2} \int_0^R \int_0^{2\pi} \left[ \sigma_{rh} (e_{r0} - \alpha(z)T(z)) \right] \cdot \sigma_{0h} (e_{00} - \alpha(z)T(z)) \cdot 2\pi r \cdot 0 \cdot dr \cdot d\theta \]

(10)

here \([e], [k], [\alpha] \) are matrix form of elastic, dielectric permeability, and piezoelectric material coefficients, respectively and are defined as

\[ [e] = \begin{bmatrix}
c_{11} & c_{12} & 0 \\
c_{21} & c_{22} & 0 \\
0 & 0 & c_{44}\end{bmatrix}, \quad [k] = \begin{bmatrix}
k_{11} & 0 & 0 \\
k_{22} & 0 & 0 \\
0 & 0 & k_{33}\end{bmatrix}, \quad [\alpha] = \begin{bmatrix}
e_1 & 0 & 0 \end{bmatrix}, \quad e_1 = e_{32}, \quad k_{11} = k_{22}, k_{33} \]

(11)

Assuming that the actuator is poled along the \( z \), and viewing the piezoelectric material as a transversely isotropic material, which is true for piezoelectric ceramics, many of the parameters in the mentioned matrices will be either zero or can be expressed in terms of the other parameters. In particular, the non-zero coefficients of piezoelectric properties may be written as

\[ c_{11} = c_{22}, c_{12} = c_{44}, \quad e_1 = e_{32}, \quad k_{11} = k_{22}, k_{33} \]

(12)

The only non-zero electric field is in the \( z \)-direction and the vector of applied electric field can be shown

\[ \{E\} = \{0 \ 0 \ Ez\}^T \]

(13)

Considering relations (4) to (10) and substituting relations (11) and integrating with respect to \( z \), the total potential energy is obtained. Applying the Euler equations for total functional of \( U \) in Eq. (11), equilibrium equations are yield and then the stability equations of the circular plate are derived using the adjacent equilibrium criterion are obtained as [4]

\[ \frac{r(N_{rr})}{2N_{rr}} + \frac{N_{0r}}{2N_{0r}} - M_{00}l_r + \frac{1}{2} N_{00}w_{1,0} + N_{01}w_{1,2} + \frac{2}{2} M_{01}l_r + (2M_{11}l_r + \frac{1}{2} M_{00})_0 = 0 \]

\[ \frac{2}{2} N_{rr} + \frac{1}{2} N_{00}l_r + \frac{1}{2} N_{00}l_r = 0 \]

(14)

The force and moment resultants of plate are expressed in terms of the stress components through the thickness as follow

\[ \{N\} = \{N_{rr} \ N_{00} \ N_{01} \}^T \]

\[ = -h/2 [\sigma]_h dz + \frac{h}{2} [\sigma]_h dz + \frac{h}{2} [\sigma]_h dz \]

\[ \{M\} = \{M_{rr} \ M_{00} \ M_{01} \}^T \]

\[ = -h/2 [\sigma]_h dz + \frac{h}{2} [\sigma]_h dz + \frac{h}{2} [\sigma]_h dz \]

(15)

Stress resultants can be simplified in the matrix form as

\[ \{N\} = \{A + h\alpha C \} \{B\} \{e(0)\} - \{N(T)\} - \{N(E)\} \]

\[ \{M\} = \{B \} \{D + L.C\} \{e(0)\} - \{M(T)\} - \{M(E)\} \]

(16)
Here \( \{N^{(v)}\}, \{M^{(v)}\}, \{N^{(v)}\}, \{M^{(v)}\} \) are the stress resultants due to the applied temperature and electrical field on the plate, and they can be computed as

\[
\{N^{(v)}\} = \{N^{(v)}\}_{\text{GM}} + \{N^{(v)}\}_{\text{MM}} \\
\{N^{(v)}\} = (e_3\varepsilon_{zh} - e_3\varepsilon_{zh} 0)^T \\
\{N^{(v)}\} = (E_4/(l-u) - E_4/(l-u) 0)^T \\
\{N^{(v)}\} = (E_1/10(c_{11} + c_{12}) - E_{10}(c_{11} + c_{12}) 0)^T \\
\{N^{(v)}\} = (E_{11}(c_{11} + c_{12}) - E_{11}(c_{11} + c_{12}) 0)^T \\
\{N^{(v)}\} = (E_{11}(c_{11} + c_{12}) - E_{11}(c_{11} + c_{12}) 0)^T \\
\{N^{(v)}\} = (E_{11}(c_{11} + c_{12}) - E_{11}(c_{11} + c_{12}) 0)^T \\
\{N^{(v)}\} = (E_{11}(c_{11} + c_{12}) - E_{11}(c_{11} + c_{12}) 0)^T
\]

where

\[
E_4 = \frac{h/2}{r^4} \alpha(z)E(2)_{Tdz} \\
E_5 = \frac{h/2}{r^4} \alpha(z)E(2)_{Tdz} \\
E_6 = \frac{h/2}{r^4} \alpha(z)E(2)_{Tdz} \\
E_7 = \frac{h/2}{r^4} \alpha(z)E(2)_{Tdz} \\
E_8 = \frac{h/2}{r^4} \alpha(z)E(2)_{Tdz} \\
E_9 = \frac{h/2}{r^4} \alpha(z)E(2)_{Tdz}
\]

\[
E_{10} = E_8 + E_9, \ E_{11} = E_6 + E_7
\]

B. Thermal Axisymmetric Buckling

Here polar symmetry condition is considered. Thus, for this case of discussion the first and second of stability equations (14), based on the displacement components, lead to

\[
(E^*_1 + h_{a}c_{11})(u_1^* - \frac{1}{r} u_1' - \frac{1}{r^2} u_1) + E^*_2(-w^*_1 - \frac{1}{r} w_1' + \frac{1}{r^2} w_1') = 0
\]

\[
(E^*_3 + 2c_{11}W^*_{11} + N_{r0}w^*_1 + \frac{1}{r} N_{00}w_1') + E^*_2(-u^*_1 - \frac{2}{r} u_1' + \frac{1}{r^2} u_1') = 0
\]

where \( E^*_1, E^*_2, E^*_3 \) are given as

\[
(E^*_1, E^*_2, E^*_3) = 1/(l - u)^2 \frac{h/2}{r^4} (1, z, z^2)E(z)dz
\]

Referring to Eqs. (16), using the membrane plate theory, the prebuckling forces are obtained as

\[
N_{\text{tr0}} = -N_{\text{tr0}} + e_{31}\varepsilon_{zh}a, \ N_{000} = -N_{000} - e_{31}\varepsilon_{zh}a
\]

Thus, the set of coupled stability equations must be solved. For clamped and immovable edge in \( r \)-direction, the boundary conditions are expressed as [5]

\[
u_1 (r = 0) = 0, \ \ w_1 (r = 0) = \text{finite}
\]

\[
u_1 (r=a) = w_1 (r=a) = w_1^* (r=a) = 0
\]

The solution of Eqs. (20) is assumed in the form

\[
u_1 (r) = A_1J_1(\lambda r) + A_2Y_1(\lambda r) + A_3(1/r) + A_4r
\]

\[
w_1 (r) = A_5J_0(\lambda r) + A_6Y_0(\lambda r) + A_7Lnr + A_8
\]

where \( J_2, J_1, Y_2, Y_1 \) are the Bessel functions of first, zero order, and first and second kinds, respectively. Also, \( A_4 \) to \( A_8 \) are the integration constants. Using the first and second boundary conditions yields \( A_3 = A_2 = A_4 = A_7 = 0 \). Satisfying the third boundary condition of Eqs. (23),

\[
A_4 = 0, \ A_8 = -J_0(\lambda a)A_5, \ -J_1(\lambda a) = 0
\]

Thus, the smallest root is \( \lambda a = 3.83 \). It is seen that for the clamped edge

\[
u_1 (r) = A_1J_1(\lambda r), \ \ w_1 (r) = A_5(J_0(\lambda r) - J_0(\lambda a))
\]

Substituting the expressions (26) into (24), two linear homogeneous equations are obtained as

\[
-\lambda^2(E^*_1 + c_{11}a)A_1 - \lambda^3E^*_2A_5 = 0
\]

\[
\lambda^3 E^*_2A_1 + [\lambda^4(E^*_3 + c_{11}L) - \lambda^2(N_{\text{tr0}} + e_{31}\varepsilon_{zh}a)]A_5 = 0
\]

For a nontrivial solution of these equations, the determinant of coefficient must be set to zero and when the temperature distribution of the plate is a function of thickness direction only, \( \lambda \) is constant and yields.

\[
\lambda^2 = (N_{\text{tr0}} + e_{31}\varepsilon_{zh})/[E^*_3 + c_{11}L - E^*_2/(E^*_1 + c_{11}a)]
\]

For the case of uniform temperature rise, taking a plate at temperature \( T_o \), and the temperature may be raised to \( T_f \) where the plate buckles. In such a case, Substituting from Eqs. (18) into Eq. (28) the critical buckling temperature \( \Delta T_{cr} \) is expressed in the form

\[
\Delta T_{cr} = (1 - u)[\lambda^2(E^*_1 + c_{11}L) - E^*_2/(E^*_1 + c_{11}a)]
\]

\[
- e_{31}\varepsilon_{zh}a] /[Q_1 + 2(c_{11} + c_{12})a\alpha p h_{a}]
\]
Where \( Q_1 = \int_{-h/2}^{h/2} \alpha(z)E(z)dz \), \( \Delta T_{cr} = T_f - T_0 \)

and \( \alpha_p \) is coefficient of thermal expansion of piezoelectric.

<table>
<thead>
<tr>
<th>Material</th>
<th>Plate and Piezoelectric</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminium</td>
<td>Alumina</td>
</tr>
<tr>
<td>( E_m = 700 \text{GPa} )</td>
<td>( E_c = 380 \text{GPa} )</td>
</tr>
<tr>
<td>( m_p = 660 \text{k})</td>
<td>( m_c = 2050 \text{k})</td>
</tr>
<tr>
<td>( \alpha_m = 23.0e-6 )</td>
<td>( \alpha_c = 7.4e-6 )</td>
</tr>
<tr>
<td>( \nu = 0.3 )</td>
<td>( \nu = 0.3 )</td>
</tr>
</tbody>
</table>

III. RESULTS AND DISCUSSION

In the following, the axisymmetric stability and thermal buckling loads of an FG circular plate integrated with piezoelectric layers subjected to uniform temperature rise is derived and summarized in the preceding section. To validate the formulations of the present article, thermal buckling loads of the circular plate are compared with those obtained by Najafizadeh and Eslami [2] for isotropic plate. It is clear that from (29), taking PFGM circular plate, the same results is obtained for the homogeneous isotropic full ceramic circular plate. The results are obtained that are identical to those reported as in [2]. Now, consider an FGM circular plate integrated with two piezoelectric layers. The material properties of piezoelectric, metal (Aluminium), and ceramic (Alumina) constituents are given in TABLE 1. For this example the results for thermal buckling loads is plotted in Figs. (2) to (4).

Figure (2) represents the critical buckling temperature versus \( h/a \) for a P-, S-, E-FGM clamped circular plate and without taking piezoelectric layers under uniform temperature rise. The mechanical boundary condition at the edge of the plate is assumed to be clamped supported. Here, the Curie temperature is an important parameter for the applications of ferroelectrics. For the PZT ceramics, the phase above Curie temperature is paraelectric and also non-piezoelectric (isotropic). If the piezoelectric properties are used in applications, the material cannot be exposed to the temperatures above the Curie temperature to preserve ferroelectric properties. It is recommended by PZT manufacturers not to use PZT ceramics above 0.5\( \theta_{c} \). The Curie temperatures \( \theta_{c} \) for commercially produced PZT’s are usually between 150 and 360 \( ^{\circ}\text{C} \)[6]. Figures (3) and (4) are part of Fig. 2 under the limit of \( \theta_{c} = 200 \text{C} \), for PFGM and SFGM plate integrated with two piezoelectric layers which is magnified. Figure (3) shows the curve associated with PFGM correspond with the data represented [3]. As it can be seen, the effect of piezoelectric layers to increase the thermal bucking temperature, is larger for circular plates with inner pure metal layer.

IV. CONCLUSIONS

In the present article, the buckling analysis of Piezoelectric FG circular plate is derive based on the classical plate theory. Boundary condition of the plate is taken to be clamped. Plate is subjected to uniform temperature rise. The thermal buckling capacity of circular plates as closed-form solution is presented. It is concluded that:

1. Voltage variation of piezoelectric layers does not have significant influence on the buckling of an. P-,S-,E-FGM plates.
2. The critical buckling temperature is reduced when volume fraction index increases, as the plate becomes more metal-rich.
3. A comparison between thermal buckling curves of power-law, sigmoid, exponential FGM circular plates in uniform temperature rise, indicate that for a certain value of \( h/a \) thermal buckling capacity of circular plate made of P-FGM is better than S-FGM.

ACKNOWLEDGMENT

The present research work was supported by Islamic Azad University - South Tehran Branch.

REFERENCES


Ahmad Reza Khorshidvand was born in Tehran, Iran, in 1969. He received PhD degree in 2012, in mechanical engineering - solid mechanics from Science and Research Branch, IAU, Tehran, IRAN. Now, he is member of faculty of mechanical engineering department, South Tehran Branch, Tehran, IRAN. His research interests include Solid Mechanics, Thermo-Elasticity, Composite Materials, and Stability.

Dr. Khorshidvand is member of IACSIT, IISRO and he has more than eight published papers in International Journals.
Fig. 2: Buckling temperature versus h/a for a P-, S-, E-FGM circular clamped plate under uniform temperature rise.

Fig 3: The effective range of piezoelectric layers versus h/a on buckling temperature a circular PFGM clamped plate.

Fig. 4: The effective range of piezoelectric layers versus h/a on buckling temperature a SFGM circular clamped plate.