Further Results on Robust Stability of Neural Networks with Multiple Time Delays

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Abstract—This paper deals with the global robust stability properties of neural networks with discrete time delays. By using a suitable Lyapunov functional, we obtain a set of delay independent sufficient conditions for the existence, uniqueness and global robust asymptotic stability of the equilibrium point. The conditions can be easily verified as they can be expressed in terms of the network parameters only. A numerical example is given to illustrate the results. The main results are shown to improve and generalize a previously published result.

Keywords: Robustness Analysis, Stability Theory, Neural Networks, Delayed Systems.

I. INTRODUCTION

In recent years, dynamical neural networks have been applied to various engineering problems such as optimization, image processing and associative memory design. In such applications, it is important to know the stability properties of the designed neural network. When employing a neural network to use an associative memory, it is required that the neural network have multiplicity of equilibrium points. On the other hand, when a neural network is designed to solve special optimization problems, then the neural network must have only one equilibrium point which is globally asymptotically stable. One may refer to [1]-[4] and the references therein for various stability results for different neural network models. In hardware implementation of neural networks, some parameters may have impact on the equilibrium and stability properties of neural networks, which are time delays occurring during the processing and transmission of the signals, and deviations in the network parameters due to the tolerances of electronic components employed in the design. In such cases, some delay parameters are introduced into the system equations that determine the dynamical behavior of neural networks and their robust stability properties. Some results concerning the robust stability properties of neural networks with delay have been reported in [5]-[15].

The activation functions are assumed to be Lipschitz continuous, i.e., there exist constants \( \ell_i > 0 \) such that
\[
|f_i(x) - f_i(y)| \leq \ell_i |x - y|, \quad i = 1, 2, ..., n, \quad \forall x, y \in R, x \neq y
\]
where \( \ell_i > 0 \) denotes a Lipschitz constant. This class of functions will be denoted by \( f \in \mathcal{L} \).

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We will need the following lemmas:

Lemma 1—14: If \( A \in [A, B] \) and \( B \in [B, \overline{B}] \), then:
\[
||A||_2 \leq ||A^*||_2 + ||A_*||_2 \quad \text{and} \quad ||B||_2 \leq ||B^*||_2 + ||B_*||_2
\]
where \( A^* = \frac{1}{2}(A + A^t), A_* = \frac{1}{2}(A - A^t), B^* = \frac{1}{2}(B + B^t), B_* = \frac{1}{2}(B - B^t) \).

Lemma 2—1: If \( H(x) \in C^0 \) satisfies the conditions

(i) \( H(x) \neq H(y) \) for all \( x \neq y \).
(ii) \( \|H(x)\| \to \infty \) as \( \|x\| \to \infty \),

then, \( H(x) \) is homeomorphism of \( \mathbb{R}^n \).

Throughout this paper, \( [v] \) will denote \( [v] = ([v_1], [v_2], ..., [v_n])^T \). For any matrix \( Q = (q_{ij})_{n \times n} \), \( \|Q\| = (\|q_{ij}\|)_{n \times n} \), and \( \lambda_m(Q) \) and \( \lambda_M(Q) \) will denote the minimum and maximum eigenvalues of \( Q \), respectively.

We will first simplify system (1) as follows: we let \( z_i(\cdot) = x_i(\cdot) - x_i^* \), and note that the \( z_i(\cdot) \) are governed by:

\[
\dot{z}_i(t) = -c_i z_i(t) + \sum_{j=1}^{n} a_{ij} g_j(z_j(t)) + h_i z_i(t - \tau_j)
\]

(4)

where \( g_i(z_i(\cdot)) = f_i(z_i(\cdot) + x_i^*) - f_i(x_i^*) \), \( i = 1, 2, ..., n \).

Note that \( f_i \), i.e., \( f \in \mathcal{L} \) implies that \( g \in \mathcal{L} \). We also note that \( g_i(0) = 0, i = 1, 2, ..., n \). System (4) can be expressed as:

\[
\dot{z}(t) = -Cz(t) + Ag(z(t)) + Bg(z(t - \tau))
\]

(5)

where \( z(t) = (z_1(t), z_2(t), ..., z_n(t))^T \in \mathbb{R}^n \) is state vector of transformed neural system, \( g(z(t)) = (g_1(z_1(t)), g_2(z_2(t)), ..., g_n(z_n(t)))^T \) and \( g(z(t - \tau)) = (g_1(z_1(t - \tau)), g_2(z_2(t - \tau)), ..., g_n(z_n(t - \tau)))^T \).

II. GLOBAL ROBUST STABILITY ANALYSIS

In this section, we present new sufficient conditions for the existence, uniqueness and global robust stability of the equilibrium point for the neural system (1). We proceed with following result:

**Theorem 1**: Let \( f \in \mathcal{L} \). Then, the neural network model (1) is globally asymptotically robust stable, if there exist positive constants \( \alpha \) and \( \beta \) such that the following condition holds:

\[
\Omega = 2C - (\alpha + \beta) - \frac{1}{\alpha}(|A^*|_2 + |A_*|_2)^2 + \frac{1}{\beta}(|B^*|_2 + |B_*|_2)^2 L^2 > 0
\]

where \( A^* = \frac{1}{2}(A + A^T), A_* = \frac{1}{2}(A - A^T), B^* = \frac{1}{2}(B + B^T), B_* = \frac{1}{2}(B - B^T) \) and \( L = diag(\ell_i > 0) \) is positive diagonal matrix.

**Proof**: We first prove the existence and uniqueness of the equilibrium point. To this end, define the following mapping associated with (1):

\[
H(x) = -Cx + Af(x) + Bf(x) + u
\]

(6)

On the other hand, if \( x^* \) is an equilibrium point of (1), then it must satisfy

\[
-Cx^* + Af(x^*) + Bf(x^*) + u = 0
\]

Since every solution of \( H(x) = 0 \) is an equilibrium point of (1), it follows from Lemma 2 that, for system defined by (1), there exists a unique equilibrium point for every input vector \( u \) if \( H(x) \) is homeomorphism of \( \mathbb{R}^n \). In the following, we will prove that \( H(x) \) is a homeomorphism of \( \mathbb{R}^n \). Let \( x, y \in \mathbb{R}^n \) be two vectors such that \( x \neq y \). For \( H(x) \) defined by (6), we can write

\[
H(x) - H(y) = -C(x - y) + A(f(x) - f(y)) + B(f(x) - f(y))
\]

(7)

We first consider the case \( x \neq y \) with \( f(x) - f(y) = 0 \); in this case, we have \( H(x) - H(y) = -C(x - y) \), where \( x - y \neq 0 \) implies that \( H(x) \neq H(y) \) since \( C \) is a positive diagonal matrix. Now consider the case where \( x - y \neq 0 \) and \( f(x) - f(y) \neq 0 \). If we multiply both sides of (7) by \( (x - y)^T \), then we get:

\[
2(x - y)^T (H(x) - H(y))
\]

\[
= -2(x - y)^T C(x - y) + 2(x - y)^T A(f(x) - f(y)) + 2(x - y)^T B(f(x) - f(y))
\]

\[
\leq -2(x - y)^T C(x - y) + \alpha(x - y)^T (x - y) + \beta(x - y)^T (x - y)
\]

\[
\leq -2(x - y)^T C(x - y) + \alpha(x - y)^T (x - y) + \beta(x - y)^T (x - y) + \frac{1}{\alpha}(|A|_2^2 (x - y)^T (x - y) + \frac{1}{\beta}(|B|_2^2 (x - y)^T (x - y)
\]

\[
\leq -2(x - y)^T C(x - y) + \alpha(x - y)^T (x - y) + \beta(x - y)^T (x - y) + \frac{1}{\alpha}(|A|_2^2 (x - y)^T (x - y) + \frac{1}{\beta}(|B|_2^2 (x - y)^T (x - y)
\]

Since \( |A|_2 \leq ||A||_2 + ||A_*||_2, ||B||_2 \leq ||B||_2 + ||B_*||_2 \), we obtain

\[
2(x - y)^T (H(x) - H(y))
\]

\[
\leq 2(x - y)^T C(x - y) + \alpha(x - y)^T (x - y) + \beta(x - y)^T (x - y) + \frac{1}{\alpha}(|A|_2^2 (x - y)^T (x - y) + \frac{1}{\beta}(|B|_2^2 (x - y)^T (x - y)
\]

\[
\Omega > 0 \] implies that \( (x - y)^T (H(x) - H(y)) < 0, \forall x \neq y \), from which it can be directly concluded that \( H(x) \neq H(y) \).
when \( x \neq y \). In order to show that \( \|H(x)\| \to \infty \) as \( \|x\| \to \infty \), we let \( y = 0 \) in (8), in which case we can derive

\[
2x^T(H(x) - H(0)) \leq -x^T \Omega x
\]

from which one can derive that \( 2\|x\|\|H(x) - H(0)\| \leq \lambda_m(\Omega) \|x\|^2 \). Using \( \|x\| \to \infty \), \( \|H(x)\| \to \infty \), and \( \|H(0)\| \to \infty \), the last inequality implies that \( \|H(x)\| \geq \lambda_m(\Omega) + 2\|H(0)\| \). Since \( \|H(0)\| \) is finite, we conclude that \( \|H(x)\| \to \infty \) as \( \|x\| \to \infty \). Hence, under the condition of Theorem 1, neural network (1) has a unique equilibrium point.

**Theorem 2:** Let \( f \in L \). Then, the neural network model (2) is globally asymptotically robust stable, if

\[
\Omega = 2\mathcal{C} - (\alpha + \beta) - \left( \frac{1}{\alpha} \langle \|A^*\|_2 + \|A_*\|_2 \rangle^2 + \frac{1}{\beta} \langle \|B^*\|_2 + \|B_*\|_2 \rangle^2 \right) L^2 > 0
\]

**Proof:** Construct the Lyapunov functional

\[
V(z(t)) = z^T(t)Cz(t) + \frac{k}{\beta} \sum_{i=1}^{n} \int_{t-t_i(t)}^{t} g_i^2(z_i(\zeta))d\zeta
\]

where \( k \) is a positive constant to be determined later. The time derivative of the functional along the trajectories of system (1) is obtained as follows

\[
\dot{V}(z(t)) = -2z^T(t)Cz(t) + 2z^T(t)A_0g(z(t)) + 2z^T(t)B_0g(z(t) - \tau(t)) + \frac{k}{\beta} \sum_{i=1}^{n} \int_{t-t_i(t)}^{t} g_i^2(z_i(\zeta))d\zeta
\]

Theorem 3—15: Assume that the activation functions are bounded and \( f \in L \). Then, the neural network model (1) is globally exponentially robust stable, if there exist positive constants \( \alpha, \beta \) and \( d_i, i = 1, 2, ..., n \) such that

\[
2d_i \ell_i - \frac{1}{\alpha} p_i - \frac{1}{\beta} q_i - (\alpha + \beta)d_i^2 \ell_i > 0, i = 1, 2, ..., n
\]

where

\[
p_i = \sum_{j=1}^{n} (a_{ij}^* \sum_{m=1}^{d_i} a_{ij}^m), \quad q_i = \sum_{j=1}^{n} (b_{ij}^* \sum_{m=1}^{d_i} b_{ij}^m), i = 1, 2, ..., n
\]

with \( a_{ij}^* = \max \{a_{ij}, |\pi_{ij}|\} \) and \( b_{ij}^* = \max \{|b_{ij}|, |\bar{b}_{ij}|\} \).

The following corollary is also given in [15]:

**Corollary 1—15:** Assume that the activation functions are bounded and \( f \in L \). Then, the neural network model (1) is globally exponentially robust stable, if there exist positive constants \( d_i, i = 1, 2, ..., n \) such that

\[
d_i \ell_i - \frac{1}{\beta} q_i - d_i^2 \ell_i > 0, \quad i = 1, 2, ..., n
\]

We will now consider the following examples:

**Example 1:** Assume that \( \mu = 0 \) and the network parameters of neural system (1) are given as follows:

\[
A = B = \begin{bmatrix} -3a & -2a \\ -a & -3a \end{bmatrix}, \quad A = \bar{A} = \begin{bmatrix} -2a & a \\ 2a & -2a \end{bmatrix},
\]

\[
C = C = C = L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

where \( a > 0 \) is real number. The matrices \( A^*, A_*, B^* \) and \( B_* \) are obtained as follows:

\[
A^* = B^* = \frac{1}{2} \begin{bmatrix} -5a & -a \\ a & -5a \end{bmatrix}, \quad A_* = B_* = \frac{1}{2} \begin{bmatrix} a & 3a \\ a & 3a \end{bmatrix}
\]

where \( \|A^*\|_2 = \|B^*\|_2 = \frac{\sqrt{26}a}{2} \) and \( \|A_*\|_2 = \|B_*\|_2 = 2a \).

Theorem 1 yields:

\[
\|A^*\|_2 + \|A_*\|_2 + \|B^*\|_2 + \|B_*\|_2 = (4 + \sqrt{26})a < 1
\]
from which the robust stability condition is obtained as $a < \frac{1}{4 + \sqrt{26}}$.

In order to check the conditions of Theorem 3, we first note that the parameters $p_1, p_2, q_1, q_2$ in Theorem 3 are calculated as follows: $p_1 = p_2 = 25a^2$ and $q_1 = q_2 = 25a^2$. Applying the result of Corollary 1 to this example, one would obtain the following inequality:

$$p_1 + q_1 = p_2 + q_2 = 50a^2 < \frac{1}{2}$$

from which the robust stability condition is determined as $a < \frac{1}{10}$. Hence, if $\frac{1}{10} \leq a < \frac{1}{4 + \sqrt{26}}$, then the result of Theorem 3 does not hold whereas the result we obtained in Theorem 1 is still applicable to this example.

REFERENCES


