Fuzzy Economic Order Quantity Model with Partial Backorder

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Abstract—In this paper, fuzzy economic order quantity (EOQ) model for inventory system with partial backorder is proposed. The fuzzy total relevance cost of the model is calculated under function principle. The optimal EOQ is derived using median rule. Fuzzy variables are appropriate when the exact information is unavailable. In the proposed model, the optimal solution for the fuzzy EOQ model is higher than the EOQ in crisp value due to the lack of information.

Keywords—fuzzy EOQ, function principle, median rule, partial backorder

I. INTRODUCTION

Inventory makes up a significant part of the total manufacturing firm asset, and plays an important role in the supply chain management [1]. For this reason, the effectiveness of the inventory management affects the profitability of a firm. A common problem in inventory system is defining the value of Economic Order Quantity (EOQ) to minimize annual inventory cost. The basic formula of EOQ is introduced by Haris [2].

\[ \text{EOQ} = \sqrt{\frac{2RC}{H}} \]  

where:

R = annual demand  
C = ordering cost  
H = annual holding cost

Classical inventory models assume deterministic parameters. However, in the real case there are many uncertainties that should be considered. Most studies on EOQ modeling use probabilistic approach [3] to deal with uncertainty. Probabilistic model assumes uncertain carrying and holding costs have certain probability distribution [4]. However, many inventory parameters uncertainties do not have any costs information. To cope with this problem, fuzzy model is used. Fuzzy model represents the uncertainty as possibility distribution, not probability distribution [4].


Some studies proposed a fuzzy EOQ with backorder. Chen et al. [8] used the function principle to simplify fuzzy operation for backorder fuzzy inventory. In their paper, they used fuzzy demand, fuzzy order cost, fuzzy inventory cost and fuzzy backorder cost. Lee and Yao [5] used triangular fuzzy number to represent a fuzzy number \( \tilde{Q} \). Centroid method is used to defuzzify \( \tilde{Q} \). In the same year, Lee and Yao [5] developed their model for backorder system. The model used trapezoidal membership function, which was proposed by Yao and Lee [9]. Kazemi et al. [10] used fuzzy parameters and decision variables for inventory system with backorders. Björk [11] proposed an analytical solution for a fuzzy EOQ. Bulančak and Kırkavak [12] applied trapezoidal fuzzy number for EOQ with backorder.

In this study, we extend the current fuzzy EOQ model to consider partial backordering. Function principle is used for fuzzy operation. Parameter uncertainty is considered for annual demand (\( R \)), order cost (\( C \)), holding cost (\( H \)), backorder cost (\( K \)) and loss cost (\( L \)). The rest of the paper is organized as follow: Section 2 describes the inventory model with partial backorder for crisp value. Section 3 explains the implementation of fuzzy set for inventory system with partial backorder. A numerical example is given in section 4. The last section provides the conclusion of this study.

II. INVENTORY MODEL WITH PARTIAL BACKORDER FOR CRISP VALUE

In this section, EOQ model with partial backorder for crisp value is reviewed. Figure 1 describes the inventory with partial backorder.

Assumptions:
1. Single product is assumed  
2. Stocks are replenished instantaneously  
3. Partial backorder is allowed
Notations used for crisp model are:

- **TRC**: total relevant cost
- **t1**: depletion period
- **t2**: backorder period
- **R**: annual demand
- **V**: maximum inventory
- **Q**: optimum order quantity
- **C**: order cost
- **H**: holding cost
- **K**: backordering cost
- **L**: loss sale cost
- **δ**: fraction of backorder that is backlogged

\[
TRC = \frac{CR}{Q} + \frac{HV^2}{2Q} + \frac{K\delta(Q-V)^2}{2Q} + \frac{L(1-\delta)(Q-V)^2}{2Q}
\]  \hspace{1cm} (8)

Optimize solution occurs when:

\[
\frac{\partial TRC}{\partial Q} = 0 \quad \text{and} \quad \frac{\partial TRC}{\partial V} = 0
\]

The result is:

\[
Q = \pm \sqrt{\frac{2CR(H + \delta K + (1-\delta)L)}{H(\delta K + (1-\delta)L)}}
\]  \hspace{1cm} (9)

As Q ≥ 0, only the positive value of Q is considered.

\[
V = \sqrt{\frac{2CR(\delta K + (1-\delta)L)}{H(H + \delta K + (1-\delta)L)}}
\]  \hspace{1cm} (10)

From (9) and (10):

If δ = 1,

\[
Q = \sqrt{\frac{2CR(H + K)}{HK}}
\]  \hspace{1cm} (11)

\[
V = \sqrt{\frac{2CRK}{H(H + K)}}
\]  \hspace{1cm} (12)

Equation (11) and (12) result in a fully backordered model. The equations are similar to EOQ model for fully backorder system in Tersine [13].

If δ = 0

\[
Q = \sqrt{\frac{2CR(H + L)}{HL}}
\]  \hspace{1cm} (13)

\[
V = \sqrt{\frac{2CRL}{H(H + L)}}
\]  \hspace{1cm} (14)

Equation (13) and (14) result in a complete lost sale model.

### III. Inventory Model with Partial Backorder For Fuzzy Number

Notations used for fuzzy model are:

- **\( \tilde{TRC} \)**: total relevant cost under fuzzy model
- **\( \tilde{H} \)**: fuzzy number for holding cost
- **\( \tilde{C} \)**: fuzzy number for ordering cost
- **\( \tilde{R} \)**: fuzzy number for annual cost
- **\( \tilde{K} \)**: fuzzy number for backordering cost
- **\( \tilde{L} \)**: fuzzy number for loss cost
- **\( \otimes \)**: Fuzzy multiplication
- **\( \overset{\Omega}{\div} \)**: Fuzzy division
⊕ = fuzzy addition

\( \tilde{Q} \) = optimum order quantity under fuzzy model

\( \tilde{V} \) = maximum inventory under fuzzy model

\( \mu_A(x) \) = the degree of value of membership function for x

w = the value of membership function

Applying Eqn. (8) for fuzzy operation, the total relevance cost using fuzzy model is described as:

\[
TRC = \frac{C \oplus R}{\tilde{Q}} + \frac{H \oplus P^2}{2 \tilde{Q}} + \frac{K \ominus D(\tilde{Q} - \tilde{V})^2}{2 \tilde{Q}} + \frac{L \ominus (1-\delta)(\tilde{Q} - \tilde{V})^2}{2 \tilde{Q}}
\]

In this study, trapezoidal is used for membership function. The trapezoidal membership function for variable \( \tilde{A} \) is describe as follow:

\[
\mu_A(x) = \begin{cases} 
  w(x-a_1)/(a_2-a_1) & \text{if } a_1 \leq x \leq a_2 \\
  w & \text{if } a_2 \leq x \leq a_3 \\
  w(x-a_4)/(a_3-a_4) & \text{if } a_3 \leq x \leq a_4 \\
  0 & \text{otherwise}
\end{cases}
\]

Where \( w < 0 \leq 1 \)

\[
\text{Figure 2. Trapezoidal membership function}
\]

Function principle is used for fuzzy operation. The trapezoidal membership function \( A \) is denoted as \((a_1, a_2, a_3, a_4; w)\) in which \( a_1 \leq a_2 \leq a_3 \leq a_4 \). Under the function principle, the fuzzy arithmetic operation for the two trapezoidal membership function \( \tilde{A} \) and \( \tilde{B} \) are described as follows [8]:

\[
\tilde{A} \oplus \tilde{B} = (a_{1} + b_{1}, a_{2} + b_{2}, a_{3} + b_{3}, a_{4} + b_{4}; w)
\]

\[
\tilde{A} \odot \tilde{B} = (a_{1}b_{1}, a_{2}b_{2}, a_{3}b_{3}, a_{4}b_{4}; w)
\]

Where \( w \) is the minimum value of \((w_1, w_2)\). Figure 4 illustrates the fuzzy arithmetic operation.

In order to defuzzify a fuzzy variable (\( \tilde{A} \)), median rule proposed by Park [14] is used. The median of \( a_m \) for \( \tilde{A} \) is derived from:

\[
\frac{\left( a_m - a_1 \right) + \left( a_m - a_2 \right)}{2} = \frac{\left( a_m - a_3 \right) + \left( a_m - a_4 \right)}{2}
\]

\[
a_m = \frac{\left( a_1 + a_2 + a_3 + a_4 \right)}{4}
\]

In this study, only the case in which \( a_1 < a_2 < a_3 < a_4 \) is considered.

\[
\text{Figure 3. Trapezoidal membership function}
\]

Based on the median rule, the total relevance cost can be formulated as:

\[
TRC = \frac{1}{4} \sum_{i=1}^{4} \left( \frac{c_r}{q_i} + \frac{h_v^2}{2q_i} + k_1^2 \delta (q_i - v_i)^2 + l_i (1 - \delta)(q_i - v_i)^2 \right)
\]

Take into account the partial derivative of \( TRC \),

\[
\frac{\partial TRC}{\partial Q} = \frac{1}{4} \sum_{i=1}^{4} \left( \frac{-c_r}{q_i^2} + \frac{h_v}{q_i} + k_1 \delta (1 - \delta)(1 - \frac{q_i}{v_i}) + l_i \delta (1 - \delta)(1 - \frac{q_i}{v_i}) \right)
\]

\[
\frac{\partial TRC}{\partial V} = \frac{1}{4} \sum_{i=1}^{4} \left( \frac{h_v}{q_i} + k_1 \delta (2v_i - 2q_i) + l_i (1 - \delta)(2v_i - 2q_i) \right)
\]

Optimal solution is obtained using the following conditions:

\[
\frac{\partial TRC}{\partial Q} = 0 \quad \text{and} \quad \frac{\partial TRC}{\partial V} = 0
\]

The results are:

\[
\tilde{Q} = \sqrt{\frac{2 \sum_{i=1}^{4} c_r h_i + \sum_{i=1}^{4} k_1 \delta + \sum_{i=1}^{4} l_i (1 - \delta)}{\sum_{i=1}^{4} h_i}}
\]

\[
\tilde{V} = \sqrt{\frac{2 \sum_{i=1}^{4} c_r h_i * \sum_{i=1}^{4} k_1 \delta + \sum_{i=1}^{4} l_i (1 - \delta)}{\sum_{i=1}^{4} h_i}}
\]
When the fuzzy membership functions consist of a single value, such that \( C = (c_1 = c_2 = c_3 = c_4 = C) \), the values of \( \tilde{Q}, \tilde{V} \) and \( TRC \) are:

\[
\tilde{Q} = \sqrt{\frac{2CR(H + \delta K + (1-\delta)L)}{(H(\delta K + (1-\delta)L)}}
\]

\[
\tilde{V} = \sqrt{\frac{2CR(\delta K + (1-\delta)L)}{H(H + \delta K + (1-\delta)L)}}
\]

\[
TRC = \frac{CR}{Q} + \frac{HV^2}{2Q} + \frac{K\delta(Q-V)^2}{2Q} + \frac{L(1-\delta)(Q-V)^2}{2Q}
\]  \( (22) \)

Equation (22) is the same as Equation (8); it means that when all the fuzzy membership consists of a single value, we can consider it as crisp value, that is \( TRC \) is the same as TRC.

IV. NUMERICAL EXAMPLE

A company needs to estimate the EOQ and total relevance cost for a product. However, the company doesn’t have exact information about the future annual demand, order cost, holding cost, backorder cost and loss sale cost. The company estimated that the annual demand \( R \) is likely to be between 6000 and 10000 units per year. The order cost \( C \) is approximately between $20 and $40 per order. The holding cost \( H \) is in the range of $2 to $4 per unit. The backorder cost \( K \) is projected to be around $5 to $10 per product. Loss sale cost \( L \) is estimated to be around $10 to $20 per unit. The value of \( \delta \) is 50%.

To solve this problem, regular trapezoidal membership functions are used to represent the uncertainty of data. The highest possible value is approximately in the mean of the range value. The highest membership value is 1. The membership functions for each possible cost are given as:

\( \tilde{R} = (6000, 7000, 9000, 10000) \)

\( \tilde{C} = (20, 25, 35, 40) \)

\( \tilde{H} = (2, 2.5, 3.5, 4) \)

\( \tilde{K} = (5, 6, 9, 10) \)

\( \tilde{L} = (10, 13, 17, 20) \)

Calculating \( TRC \):

Step 1: Calculate \( \tilde{Q} \) and \( \tilde{V} \) using (20) and (21).

\( \tilde{Q} = 461.76 \)

\( \tilde{V} = 364.55 \)

Step 2: Assigning \( \tilde{Q} \) and \( \tilde{V} \) to (19).

\( TRC = 1640.46 \)

When the variables have a single value, \( C = (c_1 = c_2 = c_3 = c_4 = C) \), it acts like a crisp value. For example, the annual demand, order cost, holding cost, backorder cost and loss sale cost values are decided from their average values, then the values become \( R = 8000, C = 30, H = 3, K = 7.5 \) and \( L = 15 \). The fuzzy membership functions become \( C = (c_1 = c_2 = c_3 = c_4 = 30) \), \( R = (r_1 = r_2 = r_3 = r_4 = 8000) \), \( H = (h_1 = h_2 = h_3 = h_4 = 3) \), \( K = (k_1 = k_2 = k_3 = k_4 = 7.5) \) and \( L = (l_1 = l_2 = l_3 = l_4 = 15) \).
Step 1: Calculate $\tilde{Q}$ and $\tilde{V}$ using (20) and (21).

$$\tilde{Q} = 450.19$$

$$\tilde{V} = 355.41$$

Step 2: Assigning $\tilde{Q}$ and $\tilde{V}$ to (19).

$$\tilde{TRC} = 1066.23$$

For the real value example, we can calculate TRC using (8).

The TRC with information is:

Step 1: Calculate Q and V using (9) and (10).

$$Q = 450.19$$

$$V = 355.41$$

Step 2: Assigning $\tilde{Q}$ and $\tilde{V}$ to (19).

$$TRC = 1066.23$$

The example shows that when the lack of information is considered, the $\tilde{TRC}$ is higher than TRC. This is a realistic result as the fuzzy $\tilde{TRC}$ considers uncertainty due to a lack of information. This can be used to estimate the contingency cost due to lack of parameters information.

V. CONCLUSION

In this study, fuzzy EOQ model for inventory with partial backorder is proposed. The ratio $\delta$ is used to represent the percentage of backordering. If $\delta = 1$, one has a complete backordering model. On the other hand, if $\delta = 0$, all backorder will be lost sale.

Fuzzy variables are appropriate when the exact information is not available. Fuzzy term is used to model uncertainty due to incomplete information. In this paper, a function principle is used in the fuzzy operation while median rule is used to find the minimization of the fuzzy variable. The fuzzy model will result in a higher cost. This is important as we consider whether to invest an additional cost on more information.