Pricing risky bond under Cox processes and multi-dimensional Levy processes

Pao-Peng Hsu¹, Ying-Hsiu Chen²

Abstract—We derive the closed-forms for risky bond under Cox processes and multi-dimensional Levy processes. There are three characteristics in our model. First, the stochastic processes are assumed by Levy processes instead of Brownian motions so that those problem including leptokurtic, volatility smile, stochastic volatility and jump are overcome. Second, derivatives positions of firm affect the intensity function of firm. Finally, the exchange rate process is incorporated into our model. It is necessary and important to price risky bond with exchange rate risk because risky bond are globally issued usually.

Keywords—Risky bond. Cox Process. Levy process. Intensity function.

I. INTRODUCTION

THERE are two approaches to modeling default risk. The structure approach, pioneered by [2] and [22] and extend by [14] in which presented an improved method of pricing vulnerable Black-Scholes options under assumptions which are appropriate in many business situations. They considered the option writer’s liabilities as pricing option though structure approach. The disadvantage of structure approach is that the firm’s asset can be difficulty and directly observed in empirical and it has difficulty in producing realistic yield spreads of short-term defaultable securities. The advantage is that the relation between default and the firm value in an explicit way can be modeled. Another approach is the reduced-form approach adopted by [5] and [20]. The default rate is assumed by a stochastic process and default rate and recovery rate are given exogenously in reduced-form approach. [13] priced credit derivative in reduced-form approach. However, the link between the structure of a firm’s asset and liabilities even economic mechanism and firm’s default risk is not clear in reduced-form approach. [15] presented a framework in which the stopping time is a Cox process and links reduced-form approach and structural approach, hence both advantages of two approaches can maintain. [21] assumed that the asset value of firm is a stochastic process and default rate depend on asset value of firm and then a Cox process can be built. The impact of capital structure changes on credit spreads can be analyzed and the closed-form of risky bond can be derived.

There are many researches to point out the fact that firms and financial institutions take derivatives positions. [9] had demonstrated the increasing usage of derivatives by financial and non-financial firms. [3] examine how risky bond affect the valuation of vulnerable options, and vice versa. Respect to the current situation of financial institutions, most of firms or financial institutions take derivatives positions and the valuations of those derivatives influence the liabilities of firms and financial institutions. Even, it causes drama major effects for assets value of firms and financial institutions, such as Lehman. In this paper, we build the Cox process to analyze this situation and to model this problem. Neither [3] nor [7] arrive at closed-form solution with economics mechanism for risky bond and vulnerable option.

[19] demonstrated that economy-wide credit risk has been rising markedly since 2007. It is very difficult to analyze the iterative influences between macroeconomic factors and financial factor such as GDP (gross domestic production) via credit spread, although it is so important. Therefore, numbers of researches study this field such that [17] studied and modeled the influences of economic indicators on the credit spreads. [12] illustrated that returns on high yield bonds have a higher correlation with equity index returns, a lower correlation with Treasury bond index returns than do low yield bonds and macro economic variables appear to influence the aggregate rate of business failures. Economic factors affecting default rates through the Cox process can also be included in intensity function.

We assume the intensity function to depend on exchange rate and interest rate. Since the interest rate is stochastic process as pricing the equity-linked notes not only the interest rate risk incorporates into Cox process but also we have to change measure when we evaluate as well as exchange rate. Another important reason we incorporate exchange rate into our model is that the equity-linked notes are often denominated in different currencies. In fact, the equity-linked notes are very global derivatives. Therefore, how to hedge exchange rate risk for investors is very importance, especially in today's significant fluctuations in U.S. dollar or the Euro dollar. Hence, our model is constructed under cross-currency.

Many empirical investigations examine the leptokurtic
feature of return on assets including index and exchange rate ([4] and [1]), volatility smile in option markets, stochastic volatility, and jump in interest rate ([15]), exchange rate and index ([8]). The pattern of smiles and spikes observed in derivative markets can be solved under stochastic volatility and jump-diffusion models and even in the case of Levy processes. We suppose the stochastic processes follow Levy process instead of Brownian motion since not only Levy process can simultaneous address these issue but also Levy process is more general than the Brownian motion. [24] considered credit spread with jump risk for valuing corporate bond, swap and credit derivatives under Cox process. Respect to past researches, index, exchange rate and interest rate are assumed by Levy processes.

This paper establishes a multi-dimensional Levy processes and Cox processes to evaluate risky bonds (RB) that hedges credit risks, exchange risks and jump risk. Since we assume the issuer’s assets depend on derivatives, hence default rate depend on the index level of the derivative’s underlying, where index (financial factor) are linked to credit risk. As we also assume multi-dimensional Levy processes, these various factors interact with each other, which makes the model a bit complicated, but since we use reduced-form approach, we can obtain the closed-form solutions of RB.

II. Model Setting

A. Default events and Hazard rate

We incorporate derivatives position into intensity function by using the approach of default event in [21]. Let the intensity process \( \phi(X_t) \) represents the approximate default probability of the firm. \( V \) denote as the asset value and \( r \) denote as the interest rate. According to [21], \( \phi(t) \) depends on \( V \) and \( r \) as follow:

\[
\phi(t)dt = \text{prob}\{ \tau \in (t,t+d) \text{ and } L \geq G \} = h(V,r)dt ,
\]

where \( L \) is loss amount and \( G \) is the equity and is a function on \( V \). We price RB under two cases. There is not the effect of derivatives in firm’s liability in Case I. The opposite is in Case II.

**Case I**: Without effect of derivatives in firm’s liability. According to [21],

\[
\phi(t) = \alpha_t + \alpha_z \ln V(t) + \alpha_r r(t) .
\]

**Case II**: With effect of derivatives in firm’s liability. With respect to Eq. (9), the hazard rate is

\[
\phi(t) = \alpha_t + \alpha_r \ln V(t) + \alpha_r (r(t) + a \ln C(t)) .
\]

Let \( \alpha_t = 0 \). Since the probability of default decrease as the asset value increase we let \( \alpha_z = -1 \).

As for the relationship between the short rate and the default rate, it may be positive or negative relationship in empirical. Therefore, we set \( \alpha_r = -1 \), which won’t harm our goal. Because derivatives position belongs to firm’s liability the valuation of derivatives increase will imply default rate increasing. It points out \( a > 0 \). Without loss general assumption, we use a European call, \( C \), instead of derivatives because such option is the building blocks for a broad class of option.

B. Multivariate Levy processes

All processes considered in what follows are defined on a common probability space \( (\Omega, \mathcal{F}_t, \mathbb{P}) \), endowed with a canonical filtration \( (\mathcal{F}_t^j)_{t \in \mathbb{R}^+} \) and associated with a Levy process \( (L(t))_{t \in \mathbb{R}^+} \), which is written in integral form:

\[
L(t) = \int_0^t c_0 dW(s) + \int_0^t \int x(\mu - \nu) (ds, dx) ,
\]

where \( W(t) \) denotes a Brownian motion, the measure \( \mu \) is a random measure of jumps in the process \( L(t) \), and \( \nu(ds, dx) \) is the \( \mathbb{P} \)-compensator of \( \mu \). We use \( L(t) \) which follows Eq. (3) to represent the \( i \)th asset dynamic process. Assume \( W^i \) and \( W^j \) are dependent Brownian motions with \( \mathbb{E}[dW^i dW^j] = \rho_{ij} dt \) and the jumps of all processes are independent. Let \( \Lambda \) and \( \chi \) be the characteristic exponent and function of \( L \), respectively. The characteristic function of a Levy process is fundamental to its construction. This allows the following Levy-Khinchin representation:

\[
\chi(z) = \mathbb{E}[e^{iz(t)}] = \exp(\int_0^t \Lambda(z) ds), \ z \in \mathbb{R} ,
\]

with

\[
\Lambda(z) = \frac{1}{2} c_0^2 z^2 + \int_0^\infty \left( e^{zx} - 1 \right) \nu(dx) , \ z \in \mathbb{R} ,
\]

The corresponding Levy process is unique in its distribution as the triplet given. Suppose \( B(t) \) is the market money account at time \( t \) and \( c' \) are constant; that is, \( c'_i = c' \) for \( s \in [0,1] \) and every \( i \). Here, the upper right \( d \) represents domestic and \( f \) represents foreign. The interest rate processes follow the mean-reversion and Levy process; thus, \( r^d(t) \) and \( r^f(t) \) can be expressed as:

\[
dr^d(t) = k^d (\frac{\theta^d}{k^d} - r^d(t)) dt + \sigma^d dL^d(t)
\]

and

\[
dr^f(t) = k^f (\frac{\theta^f}{k^f} - r^f(t)) dt + \sigma^f dL^f(t)
\]

where \( \frac{\theta^d}{k^d} \) and \( \frac{\theta^f}{k^f} \) are the domestic and foreign term-long interest rates, \( k^d \) and \( k^f \) are the domestic and foreign speeds of reversion, and \( \sigma^d \) and \( \sigma^f \) are the domestic and foreign volatilities. Let their ZCBs be formed in

\[
p^d(t,T) = p^d(0,T)B^d(t) \times \exp\left(\int_0^t -R^d(s,T) ds + \int_0^t N^d(s,T) dL^d(s)\right)
\]

and

\[
p^f(t,T) = p^f(0,T)B^f(t) \times \exp\left(\int_0^t -R^f(s,T) ds + \int_0^t N^f(s,T) dL^f(s)\right)
\]
where \( R'(t, T) = \int_t^T e^{-r_i(s)}ds \) and 
\[ N'(t, T) = \int_t^T -e^{-r_i(s)}\sigma_i'ds, \text{ for } i = d, f. \]

The exchange rate process is assumed to take the following form:
\[ d\ln e(t) = \left[r^d(t) - r^f(t) - m(t)\right]dt + \sigma_i'dL(t). \] (9)
The foreign index or stock dynamic will be:
\[ d\ln I(t) = R^i(t)dt + \sigma_i'dL^i(t). \] (10)
The value of the cash assets of firm \( i \) is:
\[ d\ln V_i(t) = R^i(t)dt + \sigma_i'dL^i(t) \] (11)
where \( i \) is firms A or B.

Note that we give the forms of drift term in Eq. (7) ~ Eq. (11) but we do not limit what they are. In what follows, \( \bar{\mu} \) represents the Levy process under domestic risk-neutral measure \( \bar{Q} \) and the drift term, which is restricted in measures \( \bar{Q} \) denoted by bar.

The prices of domestic and foreign ZCBs and the exchange rate are martingale under measure \( \bar{Q} \) as long as \( \bar{\mu}^d(t, T), \bar{\mu}^f(t, T), R'(t, T) \) satisfy the following conditions:
\[ \bar{R}(t, T) = \frac{1}{2}(\bar{N}(t, T)c^d)^2 + \int_t^T \left( e^{\bar{N}(s, T)c^d} - 1 \right)\nu^d(dx); \] (12)
\[ \bar{\mu}(t) = \frac{1}{2}(\sigma^d c^d)^2 + \int_t^T \left( e^{\nu_d} - 1 \right)\nu^d(dx); \] (13)
\[ \bar{R}'(t, T) = \frac{1}{2}(N'(t, T)c^f)^2 + \rho_{df}\sigma^d c^d N'(t, T)c^f \]
\[ + \int_t^T \left( e^{N'(s, T)c^f} - 1 \right)\nu^f(dx). \] (14)

In addition, the following price processes hold under measure \( \bar{Q} \):
\[ R'(t) = r^f(t) - \rho_{df}\sigma^d c^d c^f - \int_t^T \left( e^{\nu_d} - 1 \right)\nu^f(dx) - \frac{1}{2}(\sigma^d c^d)^2; \] (15)
\[ R'(t) = r^f(t) - \rho_{df}\sigma^d c^d c^f - \int_t^T \left( e^{\nu_d} - 1 \right)\nu^f(dx) - \frac{1}{2}(\sigma^d c^d)^2. \] (16)
\[ V_i \text{ and } I(t) \text{ under } \bar{Q} \text{ are:} \]
\[ \frac{dV_i(t)}{V_i(t)} = R^i(t)dt + \sigma_i'dL^i(t) \text{ and } \frac{dI(t)}{I(t)} = R^i(t)dt + \sigma_i'dL^i(t). \]

C. Tools for changing measure
In order to price RB, we change the numeraire and switch from measure \( Q \) to some adequate measure. In addition, the pricing formulas are expressed in terms of characteristic function. Therefore, the characteristic functions under each relevant measure will also be derived.

The foreign forward martingale measure for maturity \( T \), denoted by \( Q^f_T \), is defined by the Radon-Nikodym derivative
\[ \frac{dQ^f_T}{dQ} = p^f(T, T) / B^f(T) \]
From Eqs. (11) we have the explicit expression
\[ \frac{dQ^f_T}{dQ} = \exp(\int_0^T -R'(s, T)ds + \int_0^T N'(s, T)dL^f(s)). \]

We need the result of \( E^Q_0 \left[ \exp(\int_0^T \sigma \cdot d\bar{L}(s)) \right], z \in \mathbb{R} \)
when we price a RB. Therefore, let we define \( M^Q_T(z) = H^Q_0(T)H^Q_T(t, T) \). We have Proposition 1.

**Proposition 1:** \( \chi(z) \) satisfies Eq. (4). We then have an explicit expression for \( M^Q_T(z) \):
\[ H^Q_0(T) = \exp(\int_0^T -R'(s, T)ds)E^Q_0(\int_0^T N'(s, T)dL^f(s)) \]
and
\[ H^Q_T(t, T) = E^Q_0(\exp(\int_0^T \sigma \cdot d\bar{L}(s))). \]

Proof:
\[ M^Q_T(z) = E^Q_0(\exp(\int_0^T \sigma \cdot d\bar{L}(s))) = E^Q_0(\frac{dQ^f_T}{dQ} \exp(\int_0^T \sigma \cdot d\bar{L}(s))) \]
\[ = E^Q_0(\exp(\int_0^T -R'(s, T)ds + \int_0^T N'(s, T)dL^f(s)) \exp(\int_0^T \sigma \cdot d\bar{L}(s))) \]
\[ = \exp(\int_0^T -R'(s, T)ds) \exp(\int_0^T N'(s, T)dL^f(s)) \exp(\int_0^T \sigma \cdot d\bar{L}(s)) \]
\[ = H^Q_0(T)H^Q_T(t, T). \]

III. PRICING RISK BOND UNDER A CROSS-CURRENCY ECONOMY
Denote a foreign risk bond (RB) as \( D^f(t, T) \). Index \( I \), is underlying of the call and \( K \) is foreign strike price. In addition, we assume zero recovery to simplify our calculation. Next, we incorporate exchange rate risk into our model. It is necessary and important to price credit derivatives with exchange rate risk because they are usually issued globally. For domestic investors, the way to eliminate exchange rate risk is to have returns upon maturity denominated in the local currency, even if underlying is a foreign asset. Hence, the payoff of RB at maturity by considering exchange rate risk under cross-currency economy is:
\[ D^{-f}(T, T) = e(T)D^{f}(T, T). \] (20)

IV. THE VALUATION OF RISKY BOND
Let \( T + \tau > 0 \) and \( \tau \) is stopping time. We enlarge the original probability space, \( (\Omega, \mathcal{F}, P) \) to become \( (\Omega, \mathcal{G}, P) \) as follows:
\[ \mathcal{F}_t = \sigma\{X_s: 0 \leq s \leq t\}; \quad \mathcal{G}_t = \sigma\{1_{[s,t]}: 0 \leq s \leq t\}, \]
where \( 1 \) is an indicator function; \( G_t = \mathcal{F}_t \vee \mathcal{H}_t \).

The time \( t \) price of the foreign RB with exchange rate risk is given by:
\[ D^{-f}(t, T) = E^P\left[ e(T) \exp(-\int_t^T r^f(s)ds)1_{[t,T]}|\mathcal{G}_t \right]. \] (21)

Eq. (21) can be rewritten using the intensity function \( \phi \) as follow:
\[ D^{-f}(t, T) = 1_{[t,T]} E^P\left[ e(T) \exp(-\int_t^T (r^f(s) + \phi(s))ds) |\mathcal{G}_t \right]. \] (22)
Theorem 1. The price of a foreign RB with exchange rate risk, $D^f(t; T; \phi^f) / D^r(t; T; \phi^r)$, which the intensity function satisfy Eq. (4) / Eq. (5) issued by the firm is as follows: $D^f(t; T; \phi^f) = \mathcal{G}(T)\kappa'(T)$ and $D^r(t; T; \phi^r) = \mathcal{G}(T)\kappa''(T)$, where

$$\mathcal{G}(T) = e(t)\exp\left(\int_t^T -\bar{m}(s)ds\right)\exp\left(\int_t^T \ln V(s)ds\right)\exp\left(\int_t^T R'(y)dyds\right);$$

$$\kappa'(T) = \text{fun}([\sigma]_0)\text{fun}([\sigma]_T);$$

$$\kappa''(T) = \kappa'(T)C(t)\alpha C^m(t+1, T).$$

Here $\text{fun}[\cdot]$ represents the function on a specified characteristic function of Levy process. And restricted by $\bar{r}^f(t)$ and $\bar{p}^r(t)$ be the average values on $[t, T]$, $C^m(t+1, T)$

$$C^m(t+1, T) = \left[ e^{-\int_t^T \bar{r}^f(t)\chi dt}e^{-\int_t^T \bar{r}^r(t)\chi dt}\right]^{\alpha}$$

where $b \in \mathbb{R}$ and $mgf(-b) < \infty$, $mgf(\cdot)$ be the moment-generating function of Levy process; $\chi = \bar{r}^f(t) - \bar{r}^r(t)$ and $\alpha = \bar{r}^f(t) - \bar{r}^r(t)$.

Proof:

Suppose $E^0_t$ is conditional on measure $Q$ and filtration $\mathcal{F}_t$ and without the loss of a general assumption the RB lives before $t$. Substituting (1) into (22), we obtain:

$$E^0_t[e(T)\exp\left(\int_t^T \bar{r}^f(s)ds\right)\exp\left(\int_t^T (-\ln V(s) - \bar{r}^r(s))ds\right)]$$

$$= e(t)\exp\left(\int_t^T (-\bar{m}(s)+\int_s^T \bar{R}'(y)dyds)\exp\left(\int_t^T \ln V(0)ds\right)\right)$$

$$E^0_t[\exp\left(\int_t^T [\sigma d\bar{E}(s)+\sigma_1 \bar{L}(s)]\right)$$

$$= \mathcal{G}(t)E^0_t[\exp\left(\int_t^T [\sigma d\bar{E}(s)+\sigma_1 \bar{L}(s)]\right).$$

Under another case, $\phi^r$,

$$E^0_t[e(T)\exp\left(\int_t^T \bar{r}^r(s)ds\right)\exp\left(\int_t^T (-\ln V(s) + a\ln C(s) - \bar{r}^r(s))ds\right)$$

$$= e(t)\exp\left(\int_t^T (-\bar{m}(s)+\int_s^T \bar{R}'(y)dyds)\exp\left(\int_t^T \ln V(0)ds\right)\right)$$

$$E^0_t[\exp\left(\int_t^T [\sigma d\bar{E}(s)+\sigma_1 \bar{L}(s)]\right)E^0_t[\exp(-\int_t^T a\ln C(s)ds)]$$

$$= \mathcal{G}(t)\kappa'(t)C(t)\alpha E^0_t[\exp(-\int_t^T a\ln C(s)ds)].$$

Let $C^m(t, T) = E^0_t[\exp(-\int_{t+1}^T a\ln C(y)dy)]$. The value of the expectation for Eq. (23) is obtained when we know the characteristic exponents of the specified jump process. Hence, we denote $E^0_t[\exp\left(\int_t^T [\sigma d\bar{E}(s)]ds\right)$ as $\text{fun}([\sigma]_0)$ and $E^0_t[\exp\left(\int_t^T [\sigma d\bar{L}(s)]ds\right)$ as $\text{fun}([\sigma]_T)$, where $\text{fun}[\cdot]$ represents the function on a specified characteristic function of a Levy process. Here, the characteristic function can be expressed in closed-form if the characteristic exponent is available in closed-form.

NOTES

1. About the parameters corresponding between HJM and Vasicek model can reference [10, 11].
2. The details proof can be see [18].
3. The details of It-Levy Lemma can reference [6].

ACKNOWLEDGMENT

The authors are indebted to the National Science Council of the Republic of China, Taiwan, for their financial support of this research under Contract Nos. NSC 101-2410-H-264-006- and NSC 101-2410-H-264 -005. The authors are also most grateful for the anonymous referee’s valuable comments and constructive suggestions. Remaining errors are the responsibility of the authors.

REFERENCES


