A Numerical Hydraulic Fracture Model Using the Extended Finite Element Method

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Abstract—The hydraulic fracture phenomenon is numerically investigated in this research. This is somehow different from the classical hydraulic fracturing, which is a way of intentional creation discontinuity inside a solid; usually oil fields. The most noticeable part of the hydraulic fracture process is the effect of fluid pressure on crack surfaces which directly drives cracks towards further propagation. In this paper, the focus is on the procedure for applying the fluid traction on crack surfaces. The extended finite element method (XFEM) is employed for numerical modeling of pressurized crack problems. In contrast to the classical finite element method (FEM), considering a crack in a domain is independent from the meshing of geometry of the domain. This method allows to model different crack patterns on a fixed mesh. In this method by using the concepts of partition of unity, the standard finite element approximation is enriched with additional enrichment functions, which are closely related to the corresponding analytical solution. Its advantages facilitate the problem of modeling arbitrary cracks and discontinuities of the finite element mesh. Finally, a numerical example of pressurized crack is presented to illustrate the efficiency of the proposed approach. For verifying the results, stress intensity factors (SIFs), which are derived using the interaction integral method, are compared available reference results.

Keywords—Hydraulic fracture, Pressurized crack, Extended Finite Element Method (XFEM), Interaction integral.

I. INTRODUCTION

Among various many numerical methods which have been used for modeling the hydraulic fracture phenomenon, the extended finite element method has not been widely used in this field of study. Despite of many advantages, it has not been developed by researchers in this field for studying the hydraulic fracturing problems.

Hydraulic fracturing is a complex phenomenon in which the deformation of a material is caused by the fluid pressure on crack surfaces. First models for this phenomenon were proposed in 50’s [1, 2]. Barenblatt used the fracture mechanics principles in hydraulic fracturing problems for the first time [3, 4]. Green and Sneddon solved the elliptical hole under internal fluid pressure problem [5] and Spence and Sharp used the stress intensity factor to assess crack propagation in these problems [6].

The extended finite element method has been widely used in different crack and discontinuity problems. Motamedi and Mohammadi used this method for analyzing dynamic stationary cracks for both isotropic and orthotropic materials [7, 8]. Sukumar and Prevost utilized XFEM in quasi-static crack growth problems [9]. Cohesive crack propagation was modeled using XFEM by Belytschko and Moes [10]. In this paper, the hydraulic fracturing problem is performed in the framework of XFEM, using the interaction integral method for deriving the stress intensity factors.

II. GOVERNING EQUATIONS

Consider a body in the state of equilibrium with traction and displacement boundary condition, as shown in Fig. 1,

\[ \nabla \sigma + f^b = 0 \quad \text{in } \Omega \]  

with the following boundary conditions,

\[ \sigma \cdot n = f^t \quad \text{on } \Gamma_t \]  
\[ \sigma \cdot n = f^c \quad \text{on } \Gamma_c \]  
\[ u = \sigma^e \quad \text{on } \Gamma_u \]

Where \( \Gamma_t, \Gamma_c \) and \( \Gamma_u \) are the external traction, crack surface and displacement boundaries, respectively. \( \sigma \) is the stress tensor and \( f^b \) is the body force vector. \( f^t \) and \( f^c \) are the external and crack surface traction vectors, respectively.

The variational formulation of the boundary value problem can be defined as:
The crack surface hydrostatic traction is assumed to be perpendicular to the crack surface, as depicted in Fig. 2.

\[ W_{\text{int}} = W_{\text{ext}} \]

\[ \int_{\Omega} \sigma \cdot \delta u \, d\Omega = \int_{\Gamma_{C}} f^{+} \cdot \delta u \, d\Gamma + \int_{\Gamma_{C}} f^{-} \cdot \delta u \, d\Omega \]

\[ u^{s}(x) = \sum_{i} \phi_{i}(x) u_{i} + \sum_{j} \phi_{j}(x) H(x) a_{j} \]

\[ F_{i}(x) \]

\[ H(x) = \begin{cases} +1 & (x - x^{*}) \cdot e_{s} > 0 \\ -1 & \text{otherwise} \end{cases} \]

\[ \phi \]

\[ x^{*} \]

\[ r, \theta \]

\[ \{ F_{i}(r, \theta) \}_{i=1}^{4} = \sqrt{r} \times \left\{ \cos \theta, \sin \theta, \cos \theta \sin \theta, \sin \theta \sin \theta \right\} \]

\[ F_{i}(x) \]

Assuming a crack within an independent finite element mesh, as shown in Fig. 3, the displacement field for any point \( x \) inside the domain can be written as:

Fig. 2 Crack surface traction

Fig. 3 Arbitrary discontinuity within the FEM mesh

Fig. 4 \( x^{*} \) is the nearest point on the crack face to the point \( x \),

where \( N \) is the total number of nodes, \( N^{s} \) is the set of nodes which belong to elements which are cut by the crack (split elements), \( u_{i} \) is the vector of regular degrees of freedom, \( a_{j} \) is the vector of additional degrees of freedom related to the split elements, and \( b_{k}^{l} \) is the vector of additional degrees of freedom used for modeling the crack-tips. \( \phi \) represents the finite element shape function.

\( H(x) \) is the Heaviside enrichment function,

\( x^{*} \) is the nearest point on the crack face to the point \( x \),

where \( x \) is a Gauss integration point, as depicted in Fig. 4.
where \( N \) is the shape function of the split element and \( \sigma \) is the traction applied on the crack surfaces.

It can be shown that for an element which is cut by the crack, external nodal force vector components related to the regular degrees of freedom vanish. Only the additional degrees of freedom contribute to the external nodal force of the crack interface.

**IV. DERIVING STRESS INTENSITY FACTORS**

**A. J-Integral Concept**

For a typical cracked body, as shown in Fig. 6, the \( J \) integral can be written as,

\[
J = \int_{\Gamma_c} \left( W_s \, dy - t \frac{\partial u}{\partial x} \right) \, d\Gamma
\]

(11)

Where \( W_s \) is the strain energy density and the body force and crack surface traction are neglected. Karlsson and Backlund [12] extended the above equation to account for the effect of crack surface traction,

\[
J = \int_{\Gamma_c} \left( W_s \, dy - t \frac{\partial u}{\partial x} \right) \, d\Gamma - \int_{\Gamma_c} f_c \frac{\partial u}{\partial x} \, d\Gamma
\]

(12)

The second term must be integrated over the crack surface boundary.

**B. Equivalent Domain Integral**

For utilizing the \( J \) integral concept, a more appropriate form is usually adopted. According to the alternative method that Li et al. [13] proposed, the contour integral shown in Fig. 6 can be replaced by an equivalent area integral, as shown in Fig. 7.

\[
J = \int_{\Omega^*} \left[ \sigma_{ij} \frac{\partial u_{ij}}{\partial x_1} - W_s \delta_{ij} \right] \frac{\partial q}{\partial x_i} \, dA - \int_{\Gamma_c} f_c \frac{\partial u_{ij}}{\partial x_i} \, d\Gamma
\]

(13)

where \( q \) is a smoothing function defined as,

\[
q = \begin{cases} 
1 & \text{on } \Gamma_3 \\
0 & \text{on } \Gamma_1 
\end{cases}
\]

(14)

The value of \( q \) for any integration point can be calculated by the FEM approximation procedure.

**C. Interaction Integral Method**

This method was first proposed by Sih et al. [14]. They proposed based on this concept that the boundary value problem can be satisfied by superimposing auxiliary fields onto the actual fields. The auxiliary fields are arbitrarily chosen and are imposed in order to find a relationship between the mixed mode stress intensity factors and the interaction integrals. The contour \( J \) integral in this method can be defined as:

\[
J = J_{act} + J_{aux} + M
\]

(15)

where \( J_{act} \) and \( J_{aux} \) are associated with the actual and auxiliary states, respectively. \( M \) is defined as,

\[
M = \int_{\Omega^*} \left[ \sigma_{ij} \frac{\partial u_{ij}^{aux}}{\partial x_1} + \sigma_{ij}^{aux} \frac{\partial u_{ij}}{\partial x_1} - W_s \delta_{ij} \right] \frac{\partial q}{\partial x_j} \, dA
\]

(16)
The following equation defines the relationship of the $J$ integral and $K_I$ and $K_{II}$,

$$M = \frac{2}{E'} (K_I^{\text{aux}} + K_{II}^{\text{aux}})$$  \hspace{1cm} (18)$$

where $E' = \frac{E}{1 - \nu^2}$, for plane strain problems.

By putting $K_I^{\text{aux}} = 1$ and $K_{II}^{\text{aux}} = 0$, the mode $I$ stress intensity factor can be obtained. $K_{II}$ is obtained by setting $K_I^{\text{aux}} = 0$ and $K_{II}^{\text{aux}} = 1$.

V. NUMERICAL RESULTS

An elastic half plane containing an arbitrarily oriented internal crack near its free boundary is considered, as shown in Fig. 8. $K_I$ (Mode $I$ stress intensity factor) is computed for different crack angles. The results are compared with those presented by Erdogan and Arin [15].

Stress contours are also presented in Fig. 11, Fig. 12 and Fig.13.

Fig. 8 An elastic half plane containing an arbitrarily oriented internal crack near its free boundary

SIF results for the inclined crack are given for the value of $d/a=2$, where $a$ and $d$ are the half-length and center-to-boundary distance of the crack, respectively. Stress intensity factors are normalized by $\sigma_0 \sqrt{a}$.

In this problem, plate dimensions are assumed to be $50 \times 30$ (mm) to satisfy the infinite plate dimensions. The crack length is assumed to be 6(mm) and the crack surface traction is 1000(N/m²).

Fig. 9 and Fig.10 show the normalized stress intensity factors of right and left crack-tips, respectively.
The general observation which could be made on the basis of these results is that for the same crack surface tractions, the resistance of the medium to fracture would be higher for cracks nearly perpendicular to the boundary than for those parallel to the boundary.

V. CONCLUSION

In this paper, the extended finite element method has been utilized for modeling the hydraulic fracture phenomenon. It has been shown that XFEM makes it possible to model pressurized cracks with a satisfactory precision. The acceptable agreement between this work and available reference data shows the efficiency of the proposed method for modeling hydraulic fracture problems.

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