Output-to-State Stabilization for Takagi-Sugeno Fuzzy Neural Networks

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Abstract—This paper solves the output-to-state stabilization problem for Takagi-Sugeno fuzzy neural networks. A new set of stabilization conditions is established such that Takagi-Sugeno fuzzy neural networks are output-to-state stable and asymptotically stable simultaneously based on Riccati-type inequality. The proposed output-to-state stabilizer can be determined by solving the feasibility problem of Riccati-type inequality.

Keywords—output-to-state stabilization, Takagi-Sugeno fuzzy neural networks, Riccati-type inequality

I. INTRODUCTION

O

VER the past decade, there has been much effort to investigate dynamic neural networks due to their applications in many fields such as parallel computation, pattern recognition, function approximation, and computer vision [1]. Among several dynamic neural networks, Takagi-Sugeno fuzzy neural networks are recently accepted as an important tool to approximate several nonlinear systems. Thus, it is essential to study dynamic characteristics of Takagi-Sugeno fuzzy neural networks such as learning, stability, and performance [2, 3, 4, 5, 6, 7, 8, 9].

The output-to-state stability concept, which was introduced in [10], is a handy method to study stability or performance of several complex nonlinear systems. The output-to-state stability approach can deal with several complex nonlinear systems using only state and output information. To the best of our knowledge, there has been no published result on the output-to-state stabilization of Takagi-Sugeno fuzzy neural networks in the literature so far.

In this paper, we propose a new output-to-state stabilization method for Takagi-Sugeno fuzzy neural networks. A new set of Riccati-type inequality based conditions is established such that Takagi-Sugeno fuzzy neural networks are asymptotically stable and output-to-state stable at the same time. The gain matrix of the proposed output-to-state stabilizer can be determined by solving a set of Riccati-type inequalities [11, 12].

II. OUTPUT-TO-STATE STABILIZATION FOR TAKAGI-SUGENO FUZZY NEURAL NETWORKS

Consider the following Takagi-Sugeno fuzzy neural network:

\[ \begin{align*}
    x(t) &= \sum_{i=1}^{r} h_i(\omega)[A_i x(t) + W_i \theta(x(t)) + u(t)], \\
    y(t) &= C x(t),
\end{align*} \]

where \( x(t) = [x_1(t) \ldots x_n(t)]^T \in \mathbb{R}^r \) is the state vector, \( y(t) = [y_1(t) \ldots y_m(t)]^T \in \mathbb{R}^m \) is the output vector, \( A_i = \text{diag}[-a_{i,1}, -a_{i,2}, \ldots, -a_{i,n}] \in \mathbb{R}^{n \times n} \) is the connection weight matrix, \( W_i \in \mathbb{R}^{n \times r} (i = 1, \ldots, r) \) is the self-feedback matrix, \( \theta(x(t)) = [\theta(x(t)) \ldots \theta(x(t))]^T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is the nonlinear function vector satisfying the global Lipschitz condition with Lipschitz constant \( L_\theta > 0 \). \( C \in \mathbb{R}^{m \times n} \) is a known constant matrix, \( u(t) \in \mathbb{R}^r \) is the control input vector, \( \omega = [\omega_1, \ldots, \omega_s] \) is the premise variable, \( r \) is the number of the IF-THEN rules, \( s \) is the number of the premise variables, \( \omega = [\omega_1, \ldots, \omega_s] \), \( h_i(\omega) = w_i(\omega) / \sum_{j=1}^{s} w_j(\omega) \), \( w_i : \mathbb{R}^r \rightarrow [0, 1] \) is the membership function of the system with respect to the fuzzy rule \( i \). \( h_i \) can be regarded as the normalized weight of each IF-THEN rule and it satisfies

\[ h_i(\omega) \geq 0, \quad \sum_{i=1}^{r} h_i(\omega) = 1. \]

Now, we introduce the following definitions:

**Definition 1.** [10] A function \( \gamma : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \) is a K function if it is continuous, strictly increasing and \( \gamma(0) = 0 \). It is a \( K_n \) function if it is a \( K \) function and \( \gamma(s) \rightarrow \infty \) as \( s \rightarrow \infty \).

A function \( \beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \) is a KL function if, for each fixed \( r \geq 0 \), the function \( \beta(r, s) \) is a K function, and for each fixed \( s \geq 0 \), the function \( \beta(s, r) \) is decreasing and \( \beta(s, r) \rightarrow 0 \) as \( s \rightarrow \infty \).

In this paper, we design a stabilizer of the form

\[ u(t) = \sum_{j=1}^{r} h_j(\omega) K_j x(t) \quad (K \in \mathbb{R}^{n \times n}, \ j = 1, \ldots, r) \]

such that the Takagi-Sugeno fuzzy neural network (1) satisfies

\[ P x(t) \preceq \max \{\beta(P x(0) P t), \gamma(\sup_{0 \leq r \leq T} P y(r) P)\}, \]

where \( \gamma(s) \) is a K function and \( \beta(s, r) \) is a KL function.
In the following theorem, we design an output-to-state stabilizer for Takagi-Sugeno fuzzy neural networks:

**Theorem 1.** Assume that there exist \( P = P^T > 0 \), \( S_i = S_i^T > 0 \), \( S_2 = S_2^T > 0 \), and \( K_j \) such that
\[
A_j^T P + K_j^T P + PA_j + PK_j + L_0^2 I + P W T_i P
\]
\[-C_j^T S_j C + S_l < 0
\]
(5)

for \( i=1,\ldots,r \) and \( j=1,\ldots,r \), then the Takagi-Sugeno fuzzy neural network (1)-(2) with
\[
u(t) = \sum_{j=1}^r h_j(\omega) K_j x(t),
\]
(6)

where \( K_j \in \mathbb{R}^{m 	imes m}, j = 1, \ldots, r \), is output-to-state stable.

**Proof.** The closed-loop Takagi-Sugeno fuzzy neural network is given by
\[
\dot{x}(t) = \sum_{i=1}^r h_i(\omega) \sum_{j=1}^r h_j(\omega) [A_i + K_j] x(t)
\]
\[+ W_i \theta(x(t))
\]
(7)

Consider the Lyapunov function \( V(t) = x^T(t)Px(t) \), which satisfies the following Rayleigh inequality [13]:
\[
\lambda_{\max}(P)x(t)P \leq V(t) \leq \lambda_{\max}(P)x(t)P
\]
(8)

where \( \lambda_{\max}(\cdot) \) and \( \lambda_{\min}(\cdot) \) are the maximum and minimum eigenvalues of the matrix. The time derivative of \( V(t) \) is
\[
\dot{V}(t) = \sum_{i=1}^r h_i(\omega) \sum_{j=1}^r h_j(\omega) [x^T(t) [A_i^T P + K_j^T P
\]
\[+ PA_i + PK_j + L_0^2 I + P W_i T_i P] x(t)
\]
(9)

where
\[
\dot{V}(t) = \sum_{i=1}^r h_i(\omega) \sum_{j=1}^r h_j(\omega) [x^T(t) [A_i^T P + K_j^T P
\]
\[+ PA_i + PK_j + L_0^2 I + P W_i T_i P] x(t)
\]
(10)

If (5) is satisfied,
\[
\dot{V}(t) < \sum_{i=1}^r h_i(\omega) \sum_{j=1}^r h_j(\omega) [-x^T(t) S_j x(t)
\]
\[+ y^T(t) S_2 y(t)]
\]
(11)

\[
= -x^T(t) S_j x(t) + y^T(t) S_2 y(t)
\]
(12)

\[
\leq -\lambda_{\min}(S_j) P x(t) P^T + \lambda_{\max}(S_2) P y(t) P^T.
\]
(13)

Define
\[
\alpha_j(r) = \lambda_{\max}(P) r^2,
\]
\[
\alpha_i(r) = \lambda_{\min}(S_i) r^2,
\]
\[
\alpha_4(r) = \lambda_{\max}(S_2) r^2,
\]
where \( \alpha_j(r), \alpha_i(r), \alpha_4(r), \) and \( \alpha_4(r) \) are \( K_j \) functions. From (8) and (13), we obtain the following inequalities:
\[
\alpha_4(P x(t) P) \leq V(t) \leq \alpha_4(P x(t) P).
\]
(14)

\[
\dot{V}(t) \leq -\alpha_4(P x(t) P) + \alpha_4(P y(t) P).
\]
(15)

According to [10], \( V(t) \) is an output-to-state stable Lyapunov function. Thus, the closed-loop Takagi-Sugeno fuzzy neural network (7) satisfies (4). This completes the proof. □

By Schur complement, the condition (5) is equal to
\[
\begin{bmatrix}
Y_{i,j} & PW_i \\
W_i^T & -I
\end{bmatrix} < 0,
\]
(16)

where
\[
Y_{i,j} = A_i^T P + K_j^T P + PA_i + PK_j + L_0^2 I - C_j^T S_j C + S_l.
\]

Introducing a change of variable \( P K_j = Y_j \), the matrix inequality (16) is changed into the following linear matrix inequality (LMI):
\[
\begin{bmatrix}
\Omega_{i,j} & PW_i \\
W_i^T & -I
\end{bmatrix} < 0,
\]
(17)

where
\[
\Omega_{i,j} = (PA_i + Y_j) P + Y_i + L_0^2 I - C_j^T S_j C + S_l.
\]

Then, \( K_j = P^{-1} Y_i \).

**Corollary 1.** Assume that there exist \( P = P^T > 0 \), \( S_i = S_i^T > 0 \), \( S_2 = S_2^T > 0 \), and \( Y_j \) satisfying (17) and
\[
S_1 - C_i^T S_i C > 0.
\]
(18)

Then, the Takagi-Sugeno fuzzy neural network (1)-(2) with
\[
u(t) = \sum_{j=1}^r h_j(\omega) P^{-1} Y_j x(t)
\]
(19)

is output-to-state stable and asymptotically stable at the same time.

**Proof.** By Theorem 1, the LMI (17) ensures that the Takagi-Sugeno fuzzy neural network (1)-(2) with
\[
u(t) = \sum_{j=1}^r h_j(\omega) P^{-1} Y_j x(t)
\]
(20)

is output-to-state stable. From (12) and (2), we have
\[
\dot{V}(t) < -x^T(t)(S_1 - C_i^T S_i C)x(t)
\]
\[\leq -\lambda_{\min}(S_1 - C_i^T S_i C) P x(t) P^T.
\]
(21)

If (18) is satisfied, \( \dot{V}(t) < 0 \). Thus, we can guarantee the asymptotic stability from Lyapunov stability theory. This completes the proof. □

III. CONCLUSION

This paper has considered the output-to-state stabilization problem for Takagi-Sugeno fuzzy neural networks. A new set of sufficient LMI conditions was presented such that Takagi-Sugeno fuzzy neural networks are output-to-state stable.
and asymptotically stable simultaneously. The proposed output-to-state stabilizer can be determined by solving the feasibility problem of LMIs.

REFERENCES